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Research Article

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Constants in Discrete Poincaré and Friedrichs Inequalities and Discrete Quasi-Interpolation

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Abstract: This paper provides a discrete Poincaré inequality in n space dimensions on a simplex K with explicit constants. This inequality bounds the norm of the piecewise derivative of functions with integral mean zero on K and all integrals of jumps zero along all interior sides by its Lebesgue norm times $C(n)\operatorname{diam}(K)$. The explicit constant $C(n)$ depends only on the dimension $n = 2, 3$ in case of an adaptive triangulation with the newest vertex bisection. The second part of this paper proves the stability of an enrichment operator, which leads to the stability and approximation of a (discrete) quasi-interpolator applied in the proofs of the discrete Friedrichs inequality and discrete reliability estimate with explicit bounds on the constants in terms of the minimal angle ω_0 in the triangulation. The analysis allows the bound of two constants Λ_1 and Λ_3 in the axioms of adaptivity for the practical choice of the bulk parameter with guaranteed optimal convergence rates.

Keywords: Discrete Poincaré Inequality, Discrete Friedrichs Inequality, Enrichment Operator, Quasi-Interpolation, Discrete Reliability

MSC 2010: 65N30

1 Introduction

The first topic is the *discrete Poincaré inequality* on a simplex K with diameter h_K and a refinement $\mathcal{T}$ by newest-vertex bisection (NVB) of K. Then any compatible piecewise Sobolev function v_{NC}, such as Crouzeix–Raviart functions, with integral mean zero over K and the piecewise gradient $\nabla_{\mathrm{NC}} v_{\mathrm{NC}}$ satisfies

$$\|v_{\mathrm{NC}}\|_{L^2(K)} \le C(n) h_K \|\nabla_{\mathrm{NC}} v_{\mathrm{NC}}\|_{L^2(K)} \tag{1.1}$$

with a universal constant $C(n)$, which exclusively depends on the dimension n. This paper provides bounds of $C(n)$ for any dimension n in terms of the refinements from [10, 14] with $C(2) \le \sqrt{\frac{3}{8}}$ or $C(3) \le \frac{\sqrt{5}}{3}$ and utilizes them to prove an explicit constant in an interpolation error estimate for a discrete nonconforming interpolation operator. The discrete Poincaré inequality (1.1) is utilized, e.g., in [7, 13] without further specification of the discrete Poincaré constant.

The second topic is an enrichment operator $J_1 : \mathrm{CR}^1_0(\mathcal{T}) \to S^1_0(\mathcal{T})$ between the nonconforming and conforming P_1 finite element spaces with respect to a regular triangulation $\mathcal{T}$ into triangles for $n = 2$ with local mesh-size $h_{\mathcal{T}}$ (defined by $h_{\mathcal{T}}|_K = h_K = \operatorname{diam}(K)$ on $K \in \mathcal{T}$) and the approximation property

$$\|h_{\mathcal{T}}^{-1}(v_{\mathrm{CR}} - J_1 v_{\mathrm{CR}})\|_{L^2(\Omega)} \le c_{\mathrm{apx}} \|\nabla_{\mathrm{NC}} v_{\mathrm{CR}}\|_{L^2(\Omega)} \quad \text{for all } v_{\mathrm{CR}} \in CR^1_0(\mathcal{T}) \tag{1.2}$$

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and some global constant $c_{\mathrm{apx}} \le C(\mathcal{T})\sqrt{\cot(\omega_0)}$ for the minimal angle ω_0 in the triangulation and some topological constant $C(\mathcal{T})$ which depends only on the number of triangles that share one vertex in $\mathcal{T}$. The combination of (1.2) with an inverse estimate implies stability of J_1 with respect to the piecewise H^1 norms.

Another application of (1.2) is the *discrete Friedrichs inequality* for Crouzeix–Raviart functions

$$\|v_{\mathrm{CR}}\|_{L^2(\Omega)} \le c_{\mathrm{dF}}\|\nabla_{\mathrm{NC}} v_{\mathrm{CR}}\|_{L^2(\Omega)} \quad \text{for all } v_{\mathrm{CR}} \in CR_0^1(\mathcal{T}) \tag{1.3}$$

and some global constant c_{dF}.

The third topic is the quasi-interpolation $J := J_1 \circ I_{\mathrm{NC}} : H_0^1(\Omega) \to S_0^1(\mathcal{T})$, which combines the nonconforming interpolation operator I_{NC} with the enrichment operator J_1, and guarantees the error estimate

$$\|h_{\mathcal{T}}^{-1}(\mathrm{id} - J)v\|_{L^2(\Omega)} \le c_{\mathrm{QI}}|||v||| \quad \text{for all } v \in H_0^1(\Omega)$$

for some global constant c_{QI}. This first-order approximation property with c_{QI} and some stability constants are derived explicitly in terms of c_{apx}. A special case of this operator yields a discrete quasi-interpolation $J_{\mathrm{dQI}} : S_0^1(\hat{\mathcal{T}}) \to S_0^1(\mathcal{T})$ for a triangulation $\mathcal{T}$ with refinement $\hat{\mathcal{T}}$ such that any $\hat{v}_C \in S_0^1(\hat{\mathcal{T}})$ satisfies $\hat{v}_C = J_{\mathrm{dQI}}\hat{v}_C$ on unrefined elements $\mathcal{T} \cap \hat{\mathcal{T}}$. This enables applications to the discrete reliability, e.g., in [4] and generally in the axioms of adaptivity [2, 8] and leads to constants, which allow for a lower bound of the bulk parameter in adaptive mesh refining algorithms for guaranteed optimal convergence rates.

The remaining parts of this paper are organized as follows. The necessary notation on the triangulation and its refinements follows in Section 2 with a discrete trace identity. The discrete Poincaré inequality (1.1) is established in Section 3. The analysis provides an easy proof of the Poincaré constant in 2D for a triangle with constant $\frac{1}{\sqrt{6}}$ which is not too large in comparison with the value $\frac{1}{j_{1,1}}$ from [12] for the first positive root $j_{1,1}$ of the Bessel function of the first kind. Section 4 introduces and analyses the enrichment operator J_1 with bounds on c_{apx} in (1.2) and c_{dF} in (1.3). The quasi-interpolation follows in Section 5 and the application to discrete reliability in Section 6 concludes this paper.

The analysis of explicit constants is performed in two space dimensions for its clear geometry of a nodal patch with an easy topology. The three-dimensional analog is rather more complicated as there is no one-dimensional enumeration of all simplices, which share one vertex in a triangulation. The results are valid for higher dimension as well but the constants are less immediate to derive. The work originated from lectures on computational PDEs at the Humboldt-Universität zu Berlin over the last years to introduce students to the discrete functions spaces without a deeper introduction of Sobolev spaces.

2 Notation

For $n = 2, 3$ and any bounded Lipschitz domain $\Omega \subseteq \mathbb{R}^n$ with polyhedral boundary, let $\mathcal{T}$ denote a regular triangulation of Ω into n-simplices. Let $\mathcal{E}$ (resp. $\mathcal{E}(\Omega)$ or $\mathcal{E}(\partial\Omega)$) denote the set of all sides (resp. interior sides or boundary sides) in the triangulation and $\mathcal{N}$ (resp. $\mathcal{N}(\Omega)$ or $\mathcal{N}(\partial\Omega)$) denote the set of all nodes (resp. interior nodes or boundary nodes) in the triangulation. For any n-simplex $T \in \mathcal{T}$ with volume $|T|$, let $\mathcal{E}(T)$ denote the set of its sides (edges for $n = 2$ resp. faces for $n = 3$), $\mathcal{N}(T)$ the set of its nodes, and let $h_T := \mathrm{diam}(T)$ be its diameter. For any L^2 function $v \in L^2(\omega)$, define the integral mean $⨍_\omega v\,dx := |\omega|^{-1}\int_\omega v\,dx$ for $\omega = T \in \mathcal{T}$ or $\omega = E \in \mathcal{E}$ with surface measure $|E|$. For any node $z \in \mathcal{N}$, let $\mathcal{T}(z) := \{T \in \mathcal{T} \mid z \in \mathcal{N}(T)\}$ and $\omega_z := \bigcup_{T\in\mathcal{T}(z)} T$ the nodal patch. For $E \in \mathcal{E}$, let $\omega_E = \bigcup_{T\in\mathcal{T}, E\in\mathcal{E}(T)} T$. For $T \in \mathcal{T}$, let $\omega_T := \bigcup_{z\in\mathcal{N}(T)} \omega_z$ and let $\measuredangle(T, z)$ denote the interior angle of T at the node $z \in \mathcal{N}(T)$.

The unit normal vector ν_T along ∂T points outward. For any side $E = \partial T_+ \cap \partial T_- \in \mathcal{E}$ shared by two simplices, the enumeration of the neighboring simplices $T_\pm$ is fixed. Given any function v, define the jump of v across an inner side $E \in \mathcal{E}(\Omega)$ by $[v]_E := v|_{T_+} - v|_{T_-} \in L^2(E)$ and the jump across a boundary side $E \in \mathcal{E}(\partial\Omega)$ by $[v]_E := v$.

Definition 2.1 (Bisection). Any n-simplex $T = \mathrm{conv}\{P_1, P_2, \dots, P_{n+1}\}$ is identified with the $(n + 1)$-tuple $(P_1, P_2, \dots, P_{n+1})$. Its refinement edge is $\overline{P_1 P_{n+1}}$ and

$$\mathrm{bisec}(T) := \{T_1, T_2\}$$

is defined with $T_1 := \mathrm{conv}\{P_1, \frac{1}{2}(P_1 + P_{n+1}), P_2, \dots, P_n\}$ and $T_2 := \mathrm{conv}\{P_{n+1}, \frac{1}{2}(P_1 + P_{n+1}), P_2, \dots, P_n\}$. The ordering of the nodes in the $(n+1)$-tuples and thus, the refinement edges, for the new simplices T_1 and T_2 are fixed and for $n = 3$ additionally depend on the type of the (tagged) n-simplex [14].

Remark 2.2. There exists $M = M(n) \in \mathbb{N}$ such that any n-simplex K and

$$\mathcal{T} := \mathrm{bisec}^{(M)}(\{K\}) := \mathrm{bisec}(\mathrm{bisec}(\dots(\mathrm{bisec}(\{K\}))\dots))$$

satisfies

$$\max\{h_T \mid T \in \mathcal{T}\} \le \frac{h_K}{2}.$$

It holds that $M(2) = 3$ and $M(3) = 7$. (The latter follows from mesh-refining of the reference tetrahedron of all types [14] by undisplayed computer simulation.)

Definition 2.3. Given any initial triangulation $\mathcal{T}_0$, let $\mathbb{T} = \mathbb{T}(\mathcal{T}_0)$ be the set of all regular triangulations obtained from $\mathcal{T}_0$ with a finite number of successive bisections of appropriate simplices. For any $\mathcal{T} \in \mathbb{T}$ and $\omega \subseteq \Omega$, let $\mathcal{T}(\omega) := \{K \in \mathcal{T} \mid K \subseteq \overline{\omega}\}$. Let $\bigcup \mathbb{T}$ the set of all admissible simplices T with $T \in \mathcal{T}$ for some $\mathcal{T} \in \mathbb{T}$. The *level* of an n-simplex $T \in \bigcup \mathbb{T}$ with $T \subseteq K \in \mathcal{T}_0$ is defined as $\ell(T) := \log_2(\frac{|K|}{|T|}) \in \mathbb{N}_0$.

Remark 2.4. For any $\mathcal{T} \in \mathbb{T}$ and $T \in \bigcup \mathbb{T}$ (not necessarily $T \in \mathcal{T}$), $\mathcal{T}(T)$ satisfies exactly one of the following statements.

(a) There exists $K \in \mathcal{T}$ such that $T \subseteq K$.
(b) $\mathcal{T}(T) \in \mathbb{T}(\{T\})$, in particular, $\mathcal{T}(T)$ is a regular triangulation of T with $2 \le |\mathcal{T}(T)|$.

Definition 2.5. Define the spaces

$$H^1(\mathcal{T}) := \{v \in L^2(\Omega) \mid v|_T \in H^1(\mathrm{int}(T)) \equiv H^1(T),\ T \in \mathcal{T}\},$$
$$H^1_{\mathrm{NC}}(\mathcal{T}) := \left\{v_{\mathrm{NC}} \in H^1(\mathcal{T}) \,\middle|\, \fint_E [v_{\mathrm{NC}}]_E \,\mathrm{d}s = 0,\ E \in \mathcal{E}(\Omega)\right\}.$$

Define the discrete spaces

$$P_1(\mathcal{T}) := \{v_1 \in L^2(\Omega) \mid \text{for all } T \in \mathcal{T},\ v_1|_T \text{ is polynomial of degree } \le 1 \text{ on } T\} \subseteq H^1(\mathcal{T}),$$
$$S^1_0(\mathcal{T}) := P_1(\mathcal{T}) \cap H^1_0(\Omega) \subseteq H^1_0(\Omega),$$
$$\mathrm{CR}^1_0(\mathcal{T}) := \{v_{\mathrm{CR}} \in P_1(\mathcal{T}) \mid v_{\mathrm{CR}} \text{ continuous at } \mathrm{mid}(E),\ E \in \mathcal{E}(\Omega),\ v_{\mathrm{CR}}(\mathrm{mid}(E)) = 0,\ E \in \mathcal{E}(\partial\Omega)\} \subseteq H^1_{\mathrm{NC}}(\mathcal{T}).$$

For a function $v \in H^1(\mathcal{T})$ let $\nabla_{\mathrm{NC}} v$ denote the piecewise weak gradient and for any measurable subset $\omega \subseteq \Omega$, let $|||v|||_{\mathrm{NC}(\omega)} := \|\nabla_{\mathrm{NC}} v\|_{L^2(\omega)}$, $|||v|||_{\mathrm{NC}} := |||v|||_{\mathrm{NC}(\Omega)}$ the nonconforming energy norm and for $v \in H^1(\Omega)$, let $|||v|||_\omega := \|\nabla v\|_{L^2(\omega)}$ and $|||v||| := |||v|||_\Omega$.

A piecewise application of the Gauß divergence theorem leads to the following discrete trace identity.

Lemma 2.6 (Discrete Trace Identity). *Let $T = \mathrm{conv}\{E, P\}$ be an n-simplex with vertex $P \in \mathcal{N}(T)$ and opposite side $E \in \mathcal{E}(T)$ and $\mathcal{T}$ a regular triangulation of T. Then any $v_{\mathrm{NC}} \in H^1_{\mathrm{NC}}(\mathcal{T})$ satisfies the trace identity*

$$\fint_E v_{\mathrm{NC}} \,\mathrm{d}s = \fint_T v_{\mathrm{NC}} \,\mathrm{d}x + \frac{1}{n} \fint_T (x - P) \cdot \nabla_{\mathrm{NC}} v_{\mathrm{NC}} \,\mathrm{d}x.$$

Proof. The proof is a generalization of the continuous trace identity [6]. Let $\mathcal{E}(\mathrm{int}(T))$ the interior sides with respect to the triangulation $\mathcal{T}$. The identity

$$\mathrm{div}_{\mathrm{NC}}((\cdot - P) v_{\mathrm{NC}}) = n v_{\mathrm{NC}} + (\cdot - P) \cdot \nabla_{\mathrm{NC}} v_{\mathrm{NC}},$$

where $(\cdot - P)(x) = (x - P)$ for $x \in T$, a piecewise application of the Gauß divergence theorem, and the definition of the normal jumps $[v_{\mathrm{NC}}]_F \cdot \nu_F = v_{\mathrm{NC}}|_{T_+} \nu_{T_+} + v_{\mathrm{NC}}|_{T_-} \nu_{T_-}$ for $F = \partial T_+ \cap \partial T_-$, $T_\pm \in \mathcal{T}$, lead to

$$n \int_T v_{\mathrm{NC}} \,\mathrm{d}x + \int_T (x - P) \cdot \nabla_{\mathrm{NC}} v_{\mathrm{NC}} \,\mathrm{d}x = \sum_{F \in \mathcal{E}(\mathrm{int}(T))} \int_F [v_{\mathrm{NC}}]_F (x - P) \cdot \nu_F \,\mathrm{d}s + \sum_{F \in \mathcal{E}(T) \setminus \{E\}} \int_F v_{\mathrm{NC}} (x - P) \cdot \nu_F \,\mathrm{d}s$$
$$+ \int_E v_{\mathrm{NC}} (x - P) \cdot \nu_E \,\mathrm{d}s.$$

The observation of

$$\begin{aligned}(x-P)\cdot\nu_F &\equiv c_F \in \mathbb{R} &&\text{on any } F\in\mathcal{E}(\operatorname{int}(T)),\\ (x-P)\cdot\nu_F &\equiv 0 &&\text{on } F\in\mathcal{E}(T)\setminus\{E\},\\ (x-P)\cdot\nu_E &= \operatorname{dist}(P,E)=\frac{n|T|}{|E|} &&\text{on } E\end{aligned}$$

conclude the proof. □

Lemma 2.7. *Any n-simplex T with vertex $P\in\mathcal{N}(T)$ and the identity mapping $\cdot$ (i.e. $(\cdot-P)(x)=x-P$ for $x\in T$) satisfy*

$$\|\cdot-P\|_{L^2(T)}\le\sqrt{\frac{n}{n+2}}h_T|T|^{1/2}.$$

Proof. Let $\lambda_1,\dots,\lambda_{n+1}\in P_1(T)$ be the barycentric coordinates of the n-simplex $T=\operatorname{conv}(P_1,\dots,P_{n+1})$. Without loss of generality, assume $P=P_{n+1}=0$. The identity $x=\sum_{j=1}^{n+1}\lambda_j(x)P_j$ implies

$$\|\cdot-P\|^2_{L^2(T)}=\Big\|\sum_{j=1}^n\lambda_jP_j\Big\|^2_{L^2(T)}=\sum_{j,k=1}^n P_j\cdot P_k\int_T\lambda_j\lambda_k\,\mathrm{d}x=\Big(\sum_{j,k=1}^n P_j\cdot P_k+\sum_{j=1}^n|P_j|^2\Big)\frac{|T|}{(n+1)(n+2)}$$

with the integration formula for the barycentric coordinates

$$\int_T\lambda_j\lambda_k\,\mathrm{d}x=\frac{|T|(1+\delta_{jk})}{(n+1)(n+2)}.$$

The Cauchy inequality and $|P_j|\le h_T$ lead to the assertion. □

3 Discrete Poincaré Inequality

This section establishes a discrete Poincaré inequality on an n-simplex $K\subseteq\mathbb{R}^n$ with a constant

$$C(n)=\Big(\frac{4M(n)-3}{3n(n+2)}\Big)^{1/2}$$

with $M(n)$ from Remark 2.2 and so $C(2)=\sqrt{\frac{3}{8}}$ and $C(3)=\frac{\sqrt{5}}{3}$.

Theorem 3.1 (Discrete Poincaré Inequality). *Let K be an n-simplex and $\mathcal{T}\in\mathbb{T}(\{K\})$ be a regular triangulation of K. Then any $v_{\mathrm{NC}}\in H^1_{\mathrm{NC}}(\mathcal{T})$ satisfies*

$$\Big\|v_{\mathrm{NC}}-⨍_K v_{\mathrm{NC}}\,\mathrm{d}x\Big\|_{L^2(K)}\le C(n)h_K|||v_{\mathrm{NC}}|||_{\mathrm{NC}(K)}.$$

The proof of this theorem utilizes a distance function

$$\mathrm{d}^2(f,T):=\Big\|f-⨍_T f\,\mathrm{d}x\Big\|^2_{L^2(T)}$$

and its behavior under bisection for any $f\in L^2(T)$ in an n-simplex $T\subseteq\mathbb{R}^n$.

Lemma 3.2. *Let $\mathcal{T}\in\mathbb{T}(\{K\})$, $T\in\bigcup\mathbb{T}(\{K\})$, and $\{T_1,T_2\}=\operatorname{bisec}(T)$. Then any $v_{\mathrm{NC}}\in H^1_{\mathrm{NC}}(\mathcal{T})$ satisfies*

$$\mathrm{d}^2(v_{\mathrm{NC}},T)\le(n(n+2))^{-1}\max_{j=1,2}h^2_{T_j}|||v_{\mathrm{NC}}|||^2_{\mathrm{NC}(T)}+\sum_{j=1,2}\mathrm{d}^2(v_{\mathrm{NC}},T_j).$$

Proof. Let $F:=\partial T_1\cap\partial T_2$ and let $P_1,P_2\in\mathcal{N}(T)$ with $T_j=\operatorname{conv}\{F,P_j\}$ for $j=1,2$. Since $T\in\bigcup\mathbb{T}(\{K\})$ and $\mathcal{T}\in\mathbb{T}(\{K\})$, it holds either $T\subseteq\hat{T}\in\mathcal{T}$ for some $\hat{T}\in\mathcal{T}$ or $\mathcal{T}(T)$ is a regular triangulation of T. Hence, $v_{\mathrm{NC}}\in H^1_{\mathrm{NC}}(\mathcal{T})$ implies that $\int_F[v_{\mathrm{NC}}]_F\,\mathrm{d}s=0$ in both cases and $v_F:=⨍_F v_{\mathrm{NC}}\,\mathrm{d}s$ is well-defined. Similarly, for $j=1,2$, either

$T_j \subseteq \hat{T}_j \in \mathcal{T}$ for some $\hat{T}_j \in \mathcal{T}$ or $\mathcal{T}(T_j)$ is a regular triangulation of T_j. Therefore, it holds $v_{\mathrm{NC}}|_{T_j} \in H^1(T_j)$ or $v_{\mathrm{NC}}|_{T_j} \in H^1_{\mathrm{NC}}(\mathcal{T}(T_j))$ and thus, Lemma 2.6 is applicable on T_1 and T_2. With $\overline{v}_j := ⨍_{T_j} v_{\mathrm{NC}}\,\mathrm{d}x$ for $j = 1, 2$, the Cauchy–Schwarz inequality and Lemma 2.7 imply

$$\begin{aligned} n|\overline{v}_j - v_F| &= \left| ⨍_{T_j} (x - P_j) \cdot \nabla_{\mathrm{NC}} v_{\mathrm{NC}}\,\mathrm{d}x \right| \\ &\le \|\cdot - P_j\|_{L^2(T_j)} \frac{|||v_{\mathrm{NC}}|||_{\mathrm{NC}(T_j)}}{|T_j|} \\ &\le \frac{\sqrt{n} h_{T_j}}{\sqrt{(n+2)|T_j|}} |||v_{\mathrm{NC}}|||_{\mathrm{NC}(T_j)}. \end{aligned} \tag{3.1}$$

With $\overline{v} := ⨍_T v_{\mathrm{NC}}\,\mathrm{d}x = \frac{1}{2}(\overline{v}_1 + \overline{v}_2)$, the triangle inequality yields

$$\sum_{j=1,2} |\overline{v} - \overline{v}_j|^2 = \frac{1}{2}|\overline{v}_1 - \overline{v}_2|^2 \le \sum_{j=1,2} |\overline{v}_j - v_F|^2.$$

This, the orthogonality of $v_{\mathrm{NC}} - \overline{v}_j$ onto $\overline{v} - \overline{v}_j$ in $L^2(T_j)$, and $|T_1| = |T_2|$ show

$$\begin{aligned} \mathrm{d}^2(v_{\mathrm{NC}}, T) &= \|v_{\mathrm{NC}} - \overline{v}\|^2_{L^2(T_1)} + \|v_{\mathrm{NC}} - \overline{v}\|^2_{L^2(T_2)} \\ &= \sum_{j=1,2} \big(\|v_{\mathrm{NC}} - \overline{v}_j\|^2_{L^2(T_j)} + \|\overline{v} - \overline{v}_j\|^2_{L^2(T_j)}\big) \\ &\le \sum_{j=1,2} \big(\|v_{\mathrm{NC}} - \overline{v}_j\|^2_{L^2(T_j)} + |T_j||\overline{v}_j - v_F|^2\big). \end{aligned}$$

The combination with (3.1) concludes the proof. □

Proof of Theorem 3.1. Let $\mathcal{T}_0 := \{K\}$ and let $\mathcal{T}_\ell := \mathrm{bisec}^{(\ell)}(\mathcal{T}_0) \in \mathbb{T}(\mathcal{T}_0)$ for any $\ell \in \mathbb{N}_0$. For any multiindex $\alpha = (\alpha_1, \dots, \alpha_\ell) \in \{1, 2\}^\ell$ of length $\dim \alpha = \ell \in \mathbb{N}_0$, define the n-simplex K_α recursively by $K_\emptyset := K$ and $\{K_{(\alpha,1)}, K_{(\alpha,2)}\} = \mathrm{bisec}(K_\alpha)$ for extended multiindices $(\alpha, 1)$ and $(\alpha, 2)$ in $\{1, 2\}^{\ell+1}$. Thus, $\mathcal{T}_\ell = \{K_\alpha \mid \dim \alpha = \ell\}$ and $h_\ell := \max_{T\in\mathcal{T}_\ell} h_T$ satisfies $h_{\ell+1} \le h_\ell$. Remark 2.2 shows that any $\ell \in \mathbb{N}_0$ satisfies $h_{\ell+M} \le \frac{h_\ell}{2}$ for fixed $M = M(n) \in \mathbb{N}$, thus $h_{kM} \le h_0 2^{-k}$ for $k \in \mathbb{N}_0$. This implies

$$\sum_{\ell=0}^{\infty} h_\ell^2 = \sum_{k=0}^{\infty} \sum_{\ell=kM}^{(k+1)M-1} h_\ell^2 \le M \sum_{k=0}^{\infty} h_{kM}^2 \le M \sum_{k=0}^{\infty} h_0^2 2^{-2k} = \frac{4Mh_0^2}{3}. \tag{3.2}$$

With $\mathrm{d}_\alpha^2 := \mathrm{d}^2(v_{\mathrm{NC}}, K_\alpha)$ for any $\ell \in \mathbb{N}_0$ and any $\alpha \in \{1, 2\}^\ell$, Lemma 3.2 and the abbreviation $\gamma := (n(n+2))^{-1}$ show

$$\mathrm{d}_\alpha^2 \le \gamma h^2_{\dim\alpha+1} |||v_{\mathrm{NC}}|||^2_{\mathrm{NC}(K_\alpha)} + \sum_{j=1,2} \mathrm{d}^2_{(\alpha,j)}.$$

The sum over all multiindices of length $k \in \mathbb{N}_0$ reads

$$\sum_{\alpha\in\{1,2\}^k} \mathrm{d}_\alpha^2 \le \gamma h^2_{k+1} |||v_{\mathrm{NC}}|||^2_{\mathrm{NC}(K)} + \sum_{\beta\in\{1,2\}^{k+1}} \mathrm{d}_\beta^2.$$

Successive applications of this result and any choice of $L \ge \max_{T\in\mathcal{T}} \ell(T)$ lead to

$$\begin{aligned} \mathrm{d}_\emptyset^2 &\le \gamma h_1^2 |||v_{\mathrm{NC}}|||^2_{\mathrm{NC}(K)} + \sum_{\ell=1,2} \mathrm{d}_\ell^2 \\ &\le \gamma(h_1^2 + h_2^2) |||v_{\mathrm{NC}}|||^2_{\mathrm{NC}(K)} + \sum_{\alpha\in\{1,2\}^2} \mathrm{d}_\alpha^2 \\ &\le \dots \le \gamma\Big(\sum_{\ell=1}^{L} h_\ell^2\Big) |||v_{\mathrm{NC}}|||^2_{\mathrm{NC}(K)} + \sum_{\alpha\in\{1,2\}^L} \mathrm{d}_\alpha^2. \end{aligned} \tag{3.3}$$

Since $L \ge \max_{T\in\mathcal{T}} \ell(T)$, it follows that $\mathcal{T}_L = \mathrm{bisec}^{(L)}(K)$ is finer than $\mathcal{T}$. Therefore $v_{\mathrm{NC}}|_{K_\alpha} \in H^1(K_\alpha)$ for $\dim\alpha \ge L$ and the Poincaré inequality shows

$$\mathrm{d}_\alpha^2 = \left\| v_{\mathrm{NC}} - ⨍_{K_\alpha} v_{\mathrm{NC}}\,\mathrm{d}x \right\|^2_{L^2(K_\alpha)} \le c_{\mathrm{P}}^2 h^2_{\dim\alpha} |||v_{\mathrm{NC}}|||^2_{K_\alpha}.$$

Thus, any $L \geq \max_{T\in\mathcal{T}} \ell(T)$ satisfies

$$\sum_{\alpha\in\{1,2\}^L} \mathrm{d}_\alpha^2 \leq h_L^2 c_{\mathrm{P}}^2 |||v_{\mathrm{NC}}|||_{\mathrm{NC}(K)}^2.$$

The combination of this result with (3.2)–(3.3) yields

$$\begin{aligned}\Big\|v_{\mathrm{NC}} - \fint_K v_{\mathrm{NC}}\,\mathrm{d}x\Big\|_{L^2(K)}^2 &\leq \gamma\Big(\sum_{\ell=1}^{L} h_\ell^2\Big)|||v_{\mathrm{NC}}|||_{\mathrm{NC}(K)}^2 + h_L^2 c_{\mathrm{P}}^2 |||v_{\mathrm{NC}}|||_{\mathrm{NC}(K)}^2\\ &\leq \Big(\gamma\Big(\sum_{\ell=0}^{\infty} h_\ell^2 - h_0^2\Big) + h_L^2 c_{\mathrm{P}}^2\Big)|||v_{\mathrm{NC}}|||_{\mathrm{NC}(K)}^2\\ &\leq \Big(\frac{\gamma(4M-3)h_K^2}{3} + h_L^2 c_{\mathrm{P}}^2\Big)|||v_{\mathrm{NC}}|||_{\mathrm{NC}(K)}^2.\end{aligned}$$

The passage to the limit $L \to \infty$ and $h_L \to 0$ concludes the proof with $C(n)^2 = \frac{4M-3}{3n(n+2)}$. □

The remainder of this section is devoted to an alternative proof of the Poincaré inequality in two dimensions in the continuous case with suboptimal constant $6^{-1/2}$. The proof utilizes the techniques of the previous proof with red-refinement instead of bisection for a slightly better constant. Note that the proofs of Theorems 3.1 and 3.3 utilize only the existence of a Poincaré constant c_{P}, with neither its value nor its optimality. Compared to the optimal constant $\frac{1}{j_{1,1}} \approx 0.26$ in two dimensions [12], the suboptimal constant $6^{-1/2} \approx 0.41$ of Theorem 3.3 is competitive although it utilizes elementary tools.

Theorem 3.3 (Poincaré Inequality). *Let $K \subseteq \mathbb{R}^2$ be a triangle and $\mathcal{T} \in \mathbb{T}(K)$ a regular triangulation of K. Then any $v \in H^1(K)$ satisfies*

$$\Big\|v - \fint_K v\,\mathrm{d}x\Big\|_{L^2(K)} \leq \frac{h_K}{\sqrt{6}}|||v|||_K.$$

The proof relies on the subsequent key lemma.

Lemma 3.4. *Any $v \in H^1(K)$ in a triangle $T \subseteq K$ and its red-refinement $\{T_1, T_2, T_3, T_4\} = \mathrm{red}(T)$ satisfy*

$$\mathrm{d}^2(v, T) \leq \max_{j=1,\dots,4} h_{T_j}^2 \frac{|||v|||_T^2}{2} + \sum_{j=1}^{4} \mathrm{d}^2(v, T_j).$$

Proof. Let $F_j := \partial T_j \cap \partial T_4$ and $Q_1, Q_2, Q_3 \in \mathcal{N}(T_4)$ with $T_4 = \mathrm{conv}\{F_j, Q_j\}$ for $j = 1, \dots, 4$ as depicted in Figure 1. For $j = 1, 2, 3$, define $w_j := \fint_{F_j} v\,\mathrm{d}s$ and for $j = 1, \dots, 4$, let $\overline{v}_j := \fint_{T_j} v\,\mathrm{d}x$. Lemmas 2.6–2.7 imply, for $j = 1, 2, 3$,

$$n|\overline{v}_j - w_j| \leq \frac{\sqrt{n}h_{T_j}}{\sqrt{(n+2)|T_j|}}|||v|||_{T_j} \quad \text{and} \quad n|\overline{v}_4 - w_j| \leq \frac{\sqrt{n}h_{T_4}}{\sqrt{(n+2)|T_4|}}|||v|||_{T_4}. \tag{3.4}$$

With $\overline{v} := \fint_T v\,\mathrm{d}x = \frac{1}{4}(\sum_{j=1}^4 \overline{v}_j)$, a minimization in $\mathbb{R}$ and the weighted Young's inequality yield

$$\begin{aligned}\sum_{j=1}^{4}(\overline{v}_j - \overline{v})^2 = \min_{x\in\mathbb{R}} \sum_{j=1}^{4}(\overline{v}_j - x)^2 &\leq \sum_{j=1}^{3}(\overline{v}_j - \overline{v}_4)^2\\ &\leq \sum_{j=1}^{3} 4(\overline{v}_j - w_j)^2 + \frac{4(w_j - \overline{v}_4)^2}{3}.\end{aligned}$$

This, the orthogonality of $v - \overline{v}_j$ onto $\overline{v} - \overline{v}_j$ in $L^2(T_j)$, and $|T_1| = \dots = |T_4| = \frac{|T|}{4}$ show

$$\begin{aligned}\mathrm{d}^2(v, T) = \sum_{j=1}^{4}\|v - \overline{v}\|_{L^2(T_j)}^2 &= \sum_{j=1}^{4}\|v - \overline{v}_j\|_{L^2(T_j)}^2 + |T_j||\overline{v}_j - \overline{v}|^2\\ &\leq \sum_{j=1}^{4}\|v - \overline{v}_j\|_{L^2(T_j)}^2 + |T|\sum_{j=1}^{3}\frac{1}{4}\Big(4(\overline{v}_j - w_j)^2 + \frac{4(w_j - \overline{v}_4)^2}{3}\Big).\end{aligned}$$

The combination of this with (3.4) concludes the proof. □

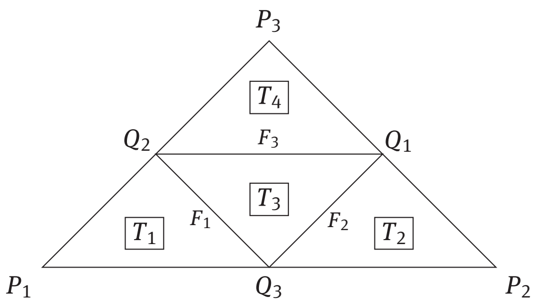

Figure 1: Red refinement of T.

Proof of Theorem 3.3. Analogously to the proof of Theorem 3.1 but with red-refinement instead of bisection, let $\mathcal{T}_0 := \{K\}$ and $\mathcal{T}_\ell := \operatorname{red}^{(\ell)}(\mathcal{T}_0) \in \mathbb{T}(\mathcal{T}_0)$ for any $\ell \in \mathbb{N}_0$. For any multiindex $\alpha = (\alpha_1, \dots, \alpha_\ell) \in \{1, \dots, 4\}^\ell$ of length $\dim \alpha = \ell \in \mathbb{N}_0$, define the n-simplex K_α recursively by $K_\emptyset := K$ and $\{K_{(\alpha,1)}, \dots, K_{(\alpha,4)}\} = \operatorname{red}(K_\alpha)$ for extended multiindices $(\alpha, 1), \dots, (\alpha, 4)$ in $\{1, \dots, 4\}^{l+1}$. Thus, $\mathcal{T}_\ell = \{K_\alpha \mid \dim \alpha = \ell\}$ and $h_\ell := \max_{T\in\mathcal{T}_\ell} h_T$ satisfies $h_{\ell+1} \le \frac{h_\ell}{2}$. Consequently,

$$\sum_{\ell=0}^{\infty} h_\ell^2 \le h_0^2 \sum_{\ell=0}^{\infty} 4^{-\ell} = \frac{4h_0^2}{3}.$$

Successive applications of Lemma 3.4 as in the proof of Theorem 3.1 lead to

$$\begin{aligned}\Big\| v - \fint_K v \,\mathrm{d}x \Big\|_{L^2(K)} &\le \Big(\sum_{\ell=1}^{L} h_\ell^2 \Big) \frac{|||v|||_K^2}{2} + h_L^2 c_{\mathrm{P}}^2 |||v|||_K^2 \\ &\le \Big(\frac{h_K^2}{6} + h_L^2 c_{\mathrm{P}}^2 \Big) |||v|||_K^2.\end{aligned}$$

The passage to the limit as $L \to \infty$ and $h_L \to 0$ concludes the proof. □

The following theorem utilizes the discrete Poincaré inequality to prove a generalization of the error estimate for nonconforming interpolation [3] to nonconforming functions and also for $n = 3$.

Theorem 3.5 (Discrete Nonconforming Interpolation). *Set* $\kappa_{\mathrm{NC}}^2 := C^2(n) + (n+1)^{-1}(n+2)^{-1}n^{-2}$. *Furthermore, let* $I_{\mathrm{NC}}\hat{v}_{\mathrm{CR}} \in \mathrm{CR}_0^1(\mathcal{T})$ *with* $(I_{\mathrm{NC}}\hat{v}_{\mathrm{CR}})(\operatorname{mid}(E)) = \fint_E \hat{v}_{\mathrm{CR}} \,\mathrm{d}s$ *for all* $E \in \mathcal{E}$ *denote the nonconforming interpolation of the Crouzeix–Raviart function* $\hat{v}_{\mathrm{CR}} \in CR_0^1(\hat{\mathcal{T}})$ *on the refinement* $\hat{\mathcal{T}} \in \mathbb{T}(\mathcal{T})$ *of* $\mathcal{T}$. *Then*

$$h_K^{-1} \|\hat{v}_{\mathrm{CR}} - I_{\mathrm{NC}}\hat{v}_{\mathrm{CR}}\|_{L^2(K)} \le \kappa_{\mathrm{NC}} |||\hat{v}_{\mathrm{CR}} - I_{\mathrm{NC}}\hat{v}_{\mathrm{CR}}|||_{\mathrm{NC}(K)} \quad \textit{for any } K \in \mathcal{T}.$$

Proof. Let $M = \operatorname{mid}(K)$, let $\mathcal{E}(K) = \{E_1, \dots, E_{n+1}\}$, and let $T_j = \operatorname{conv}\{E_j, M\}$ for $j = 1, \dots, n+1$. Then it holds that $\hat{w}_{\mathrm{CR}} := (\hat{v}_{\mathrm{CR}} - I_{\mathrm{NC}}\hat{v}_{\mathrm{CR}})|_K \in H_{\mathrm{NC}}^1(\hat{\mathcal{T}}(K))$ satisfies $\int_{E_j} \hat{w}_{\mathrm{CR}} \,\mathrm{d}s = 0$ and so Lemma 2.6 shows

$$\hat{w}_K |K| := \int_K \hat{w}_{\mathrm{CR}} \,\mathrm{d}x = \sum_{j=1}^{n+1} \int_{T_j} \hat{w}_{\mathrm{CR}} \,\mathrm{d}x = \frac{1}{n} \int_K (M - x) \cdot \nabla_{\mathrm{NC}} \hat{w}_{\mathrm{CR}} \,\mathrm{d}x.$$

This and the discrete Poincaré inequality prove

$$\begin{aligned}\|\hat{w}_{\mathrm{CR}}\|_{L^2(K)}^2 &= \|\hat{w}_{\mathrm{CR}} - \hat{w}_K\|_{L^2(K)}^2 + |K||\hat{w}_K|^2 \\ &\le C^2(n) h_K^2 |||\hat{w}_{\mathrm{CR}}|||_{\mathrm{NC}(K)}^2 + n^{-2}|K|^{-1} |||\hat{w}_{\mathrm{CR}}|||_{\mathrm{NC}(K)}^2 \|\cdot - M\|_{L^2(K)}^2.\end{aligned}$$

A modification in the proof of Lemma 2.7 with $M = 0$ and therefore $\sum_{j,k=1}^{n+1} P_j \cdot P_k = 0$ proves

$$\|\cdot - M\|_{L^2(K)}^2 \le \frac{h_K^2 |K|}{(n+1)(n+2)}.$$

This concludes the proof. □

4 Enrichment Operator

This section contains an interpolation estimate for a discrete interpolation operator $J_C : \mathrm{CR}^1_0(\mathcal{T}) \to S^1_0(\mathcal{T})$ and the discrete Friedrichs inequality. Throughout this section, consider $n = 2$.

Remark 4.1 (Three-Dimensional Case). The techniques of this section apply to the three-dimensional case as well, but lead to more complicated constants and are not minutely detailed for brevity. The point is that there is no elementary enumeration of all simplices in a nodal patch. Therefore, the examination of different configurations leads to an eigenvalue problem with constants depending on the shape of the simplices.

Lemma 4.2. *For any $2 \le J \in \mathbb{N}$ and $x \in \mathbb{R}^J$, let $x_{J+1} := x_1$, $\min x := \min\{x_1, \dots, x_J\}$, $\max x := \max\{x_1, \dots, x_J\}$. Then it holds*

$$\max_{x\in\mathbb{R}^J\setminus\{0\},\, \min x\le 0\le \max x} \frac{|x|^2}{\sum_{j=1}^J (x_{j+1}-x_j)^2} = \max_{y\in\mathbb{R}^J\setminus\{0\}} \frac{|y|^2}{\sum_{j=1}^J (y_{j+1}-y_j)^2 + (y_1+y_J)^2} = \frac{1}{2(1-\cos(\frac{\pi}{J}))}.$$

Proof. Define

$$K_1 := \{x \in \mathbb{R}^J \setminus \{0\} \mid \min x \le 0 \le \max x\},$$
$$K_2 := \{x \in \mathbb{R}^J \setminus \{0\} \mid \min x = 0\},$$
$$K_3 := \{x \in \mathbb{R}^J \setminus \{0\} \mid x_1 = 0\}.$$

For $x \in K_1$ and $\min x \le \mu \le \max x$, $y := (x_j - \mu)_{j=1,\dots,J} \in K_1$ and

$$\sum_{j=1}^J y_j^2 = \sum_{j=1}^J x_j^2 - 2\mu \sum_{j=1}^J x_j + \mu^2 J.$$

This quadratic function of μ attains its maximum at $\min x$ or $\max x$; then

$$\frac{|x|^2}{\sum_{j=1}^J (x_{j+1}-x_j)^2} \le \frac{\max\{|x-\min x|^2, |x-\max x|^2\}}{\sum_{j=1}^J (x_{j+1}-x_j)^2}.$$

Consequently, $(x - \min x)$, $-(x - \max x) \in K_2$ and the permutability of the indices show that

$$\max_{x\in K_1} \frac{|x|^2}{\sum_{j=1}^J (x_{j+1}-x_j)^2} = \max_{x\in K_2} \frac{|x|^2}{\sum_{j=1}^J (x_{j+1}-x_j)^2} = \max_{x\in K_3} \frac{|x|^2}{\sum_{j=1}^J (x_{j+1}-x_j)^2}.$$

Furthermore, any $x \in K_3$ satisfies $\sum_{j=1}^J (x_{j+1}-x_j)^2 = x_2^2 + \sum_{j=2}^{J-1}(x_{j+1}-x_j)^2 + x_J^2 = \tilde{x} \cdot A\tilde{x}$ with $\tilde{x} = (x_2, \dots, x_J)$ and the tridiagonal $(J-1) \times (J-1)$ matrix

$$A = \begin{pmatrix} 2 & -1 & & \\ -1 & \ddots & \ddots & \\ & \ddots & \ddots & -1 \\ & & -1 & 2 \end{pmatrix} \in \mathbb{R}^{(J-1)\times(J-1)}.$$

A direct calculation with the trigonometric addition formulas for the sine function shows that for any $k = 1, \dots, J-1$, the vector x^k with components

$$x_j^k = \sin\Big(\frac{kj\pi}{J}\Big)$$

is an eigenvector of A with eigenvalue

$$\lambda_k := 2\Big(1 - \cos\Big(\frac{k\pi}{J}\Big)\Big) > 0,$$

see [15, Theorem 3.2 (v)]. Since $0 < \lambda_1 < \cdots < \lambda_{J-1}$, A is positive definite and $\lambda_1|x|^2 = \lambda_1|\tilde{x}|^2 \le \tilde{x} \cdot A\tilde{x}$ concludes the proof of the first equality.

For the second equality, observe that any $y \in \mathbb{R}^J \setminus \{0\}$ satisfies $\sum_{j=1}^{J}(y_{j+1} - y_j)^2 + (y_1 + y_J)^2 = y \cdot By$ with the tridiagonal matrix

$$B = \begin{pmatrix} 3 & -1 & & & \\ -1 & 2 & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & 2 & -1 \\ & & & -1 & 3 \end{pmatrix} \in \mathbb{R}^{J\times J}.$$

A straight-forward calculation shows that the vectors $y^k \in \mathbb{R}^J$ with components

$$y_j^k = \Big(1 + \cos\Big(\frac{k\pi}{J}\Big)\Big)\sin\Big(\frac{kj\pi}{J}\Big) - \sin\Big(\frac{k\pi}{J}\Big)\cos\Big(\frac{kj\pi}{J}\Big) \quad \text{for } k = 1, \dots, J-1$$

and

$$y_j^J = \cos\Big(\frac{j\pi}{J}\Big)$$

are eigenvectors of B with eigenvalues $\lambda_k := 2(1 - \cos(\frac{k\pi}{J})) > 0$ for $k = 1, \dots, J$, see [15, Theorem 3.4 (iii)]. Consequently, $\lambda_1 |y|^2 \le y \cdot By$. □

Let $v_{\mathrm{CR}} \in \mathrm{CR}_0^1(\mathcal{T})$ and $v_{\mathrm{C}} := J_{\mathrm{C}}(v_{\mathrm{CR}}) \in S_0^1(\mathcal{T})$ with

$$v_{\mathrm{C}}(z) \in \operatorname{conv}\{v_{\mathrm{CR}}|_T(z) \mid T \in \mathcal{T}(z)\} \quad \text{for any } z \in \mathcal{N}(\Omega). \tag{4.1}$$

The shape regularity of $\mathcal{T}$ leads to a minimum angle ω_0 in $\mathcal{T}$, i.e. $0 < \omega_0 \le \min \sphericalangle\mathcal{T}$. Let

$$\begin{aligned} M_{\mathrm{int}} &:= \max\{|\mathcal{T}(z)| \mid z \in \mathcal{N}(\Omega)\} \ge 2,\\ M_{\mathrm{bd}} &:= \max\{|\mathcal{T}(z)| \mid z \in \mathcal{N}(\partial\Omega)\},\\ M_{\mathrm{patch}} &:= \max\{M_{\mathrm{int}}, M_{\mathrm{bd}}\}, \end{aligned}$$

and define

$$c_{\mathrm{apx}}^2 = \frac{\sqrt{3}\cot(\omega_0)}{2 - 2\cos(\frac{\pi}{M_{\mathrm{patch}}})}.$$

Remark 4.3. The estimate $(1 - \cos(x))^{-1} \le \frac{4}{x^2}$ for $0 < x \le \frac{\pi}{2}$ leads to the simpler estimate

$$c_{\mathrm{apx}} \le \frac{(2\sqrt{3}\cot(\omega_0))^{1/2} M_{\mathrm{patch}}}{\pi}.$$

Remark 4.4. For the case of a triangulation with right isosceles triangles,

$$c_{\mathrm{apx}} = \Big(\frac{\sqrt{3}}{2 - 2\cos(\frac{\pi}{8})}\Big)^{1/2} \le 3.3729.$$

Theorem 4.5 (Interpolation Error for J_{C}). *Any interpolation operator $J_{\mathrm{C}} : \mathrm{CR}_0^1(\mathcal{T}) \to S_0^1(\mathcal{T})$ with (4.1) satisfies*

$$\|h_{\mathcal{T}}^{-1}(1 - J_{\mathrm{C}})v_{\mathrm{CR}}\|_{L^2(\Omega)} \le c_{\mathrm{apx}} |||v_{\mathrm{CR}}|||_{\mathrm{NC}}.$$

This estimate also holds for any $T \in \mathcal{T}$ in that

$$\|h_T^{-1}(1 - J_{\mathrm{C}})v_{\mathrm{CR}}\|_{L^2(T)} \le c_{\mathrm{apx}} |||v_{\mathrm{CR}}|||_{\mathrm{NC}(\omega_T)}.$$

Proof. For any $T \in \mathcal{T}$ and $z \in \mathcal{N}(T)$, let

$$e_T(z) := v_{\mathrm{CR}}|_T(z) - v_{\mathrm{C}}(z) \quad \text{and} \quad e(z)^2 := \sum_{T\in\mathcal{T}(z)} e_T(z)^2.$$

With $e_T := (e_T(z))_{z\in\mathcal{N}(T)} \in \mathbb{R}^3$, a direct calculation with mass matrix

$$M = \frac{|T|}{12}\begin{pmatrix} 2 & 1 & 1\\ 1 & 2 & 1\\ 1 & 1 & 2 \end{pmatrix} \in \mathbb{R}^{3\times 3}$$

of the barycentric coordinates with eigenvalues $\frac{|T|}{12}$ and $\frac{|T|}{3}$ and the estimate $|T| \le \frac{\sqrt{3}h_T^2}{4}$ shows

$$h_T^{-2}\|v_{\mathrm{CR}} - v_{\mathrm{C}}\|_{L^2(T)}^2 = h_T^{-2} e_T \cdot M e_T \le \frac{|T|}{3h_T^2}|e_T|^2 \le \frac{1}{4\sqrt{3}} \sum_{z\in\mathcal{N}(T)} e_T(z)^2. \tag{4.2}$$

Any $T \in \mathcal{T}$ and $p_1 \in P_1(T)$ satisfy

$$\max_{z_1,z_2\in\mathcal{N}(T)} |p_1(z_1) - p_1(z_2)|^2 \le \frac{h_T^2 |||p_1|||_T^2}{|T|}.$$

This, $\frac{h_T^2}{|T|} \le 4\cot(\omega_0)$ and the triangle inequality show that any $\partial T_+ \cap \partial T_- \in \mathcal{E}(\Omega)$ with $z \in \mathcal{N}(E)$ and $T_\pm \in \mathcal{T}$ satisfies

$$\begin{aligned}
|e_{T_+}(z) - e_{T_-}(z)| &= |v_{\mathrm{CR}}|_{T_+}(z) - v_{\mathrm{CR}}|_{T_-}(z)| \\
&\le |v_{\mathrm{CR}}|_{T_+}(z) - v_{\mathrm{CR}}(\mathrm{mid}(E))| + |v_{\mathrm{CR}}(\mathrm{mid}(E)) - v_{\mathrm{CR}}|_{T_-}(z)| \\
&\le \frac{1}{2}\max_{z_1,z_2\in\mathcal{N}(T_+)} |v_{\mathrm{CR}}|_{T_+}(z_1) - v_{\mathrm{CR}}|_{T_+}(z_2)| + \frac{1}{2}\max_{z_1,z_2\in\mathcal{N}(T_-)} |v_{\mathrm{CR}}|_{T_-}(z_1) - v_{\mathrm{CR}}|_{T_-}(z_2)| \\
&\le \cot(\omega_0)^{1/2}(|||v_{\mathrm{CR}}|||_{T_+} + |||v_{\mathrm{CR}}|||_{T_-}) \\
&\le (2\cot(\omega_0))^{1/2} |||v_{\mathrm{CR}}|||_{\mathrm{NC}(\omega_E)}.
\end{aligned} \tag{4.3}$$

Analogously, $E \in \mathcal{E}(\partial\Omega)$ with $T \in \mathcal{T}$, $E \in \mathcal{E}(T)$, and $z \in \mathcal{N}(E)$ satisfies $|e_T(z)| \le \cot(\omega_0)^{1/2}|||v_{\mathrm{CR}}|||_{\mathrm{NC}(T)}$.

Consider $z \in \mathcal{N}(\partial\Omega)$ with $\mathcal{T}(z) = \{T_1, \dots, T_J\}$ and let

$$E_1 := \partial T_1 \cap \partial\Omega, \qquad E_{J+1} := \partial T_J \cap \partial\Omega, \qquad E_{j+1} := \partial T_j \cap \partial T_{j+1} \in \mathcal{E}(\Omega) \quad \text{for } j = 1, \dots, J-1.$$

With $e_j := e_{T_j}(z)$ for $j = 1, \dots, J$ and $e_{J+1} := e_1$, the previous estimates show that $|e_j|^2 \le \cot(\omega_0)|||v_{\mathrm{CR}}|||_{\mathrm{NC}(T_j)}^2$ for $j = 1, J$ and $|e_j - e_{j+1}|^2 \le 2\cot(\omega_0)|||v_{\mathrm{CR}}|||_{\mathrm{NC}(\omega_{E_{j+1}})}^2$ for $j = 1, \dots, J-1$. Hence

$$|e_1 + e_J|^2 + \sum_{j=1}^{J} |e_{j+1} - e_j|^2 = 2|e_1|^2 + \sum_{j=1}^{J-1} |e_{j+1} - e_j|^2 + 2|e_J|^2 \le 4\cot(\omega_0)|||v_{\mathrm{CR}}|||_{\mathrm{NC}(\omega_z)}^2.$$

This and Lemma 4.2 show that $e = (e_1, \dots, e_J)^\top \in \mathbb{R}^J$ satisfies

$$e(z)^2 = |e|^2 \le \frac{2\cot(\omega_0)}{1 - \cos(\frac{\pi}{J})} |||v_{\mathrm{CR}}|||_{\mathrm{NC}(\omega_z)}^2.$$

Let $z \in \mathcal{N}(\Omega)$ with $\mathcal{T}(z) = \{T_1, \dots, T_J\}$ and

$$T_{J+1} := T_1, \qquad \partial T_J \cap \partial T_1 \in \mathcal{E}(\Omega), \qquad \partial T_j \cap \partial T_{j+1} \in \mathcal{E}(\Omega) \quad \text{for } j = 1, \dots, J-1.$$

Then (4.3) shows that $|e_j - e_{j+1}|^2 \le 2\cot(\omega_0)|||v_{\mathrm{CR}}|||_{\mathrm{NC}(T_j \cup T_{j+1})}^2$ for $j = 1, \dots, J$. Since $0 \in \mathrm{conv}\{e_1, \dots, e_J\}$, it follows that $\min e \le 0 \le \max e$ and Lemma 4.2 leads to

$$e(z)^2 = |e|^2 \le \frac{2\cot(\omega_0)}{1 - \cos(\frac{\pi}{J})} |||v_{\mathrm{CR}}|||_{\mathrm{NC}(\omega_z)}^2. \tag{4.4}$$

Altogether, any $z \in \mathcal{N}$ satisfies

$$e(z)^2 \le \frac{2\cot(\omega_0)}{1 - \cos(\frac{\pi}{M_{\mathrm{patch}}})} |||v_{\mathrm{CR}}|||_{\mathrm{NC}(\omega_z)}^2 =: \frac{4}{\sqrt{3}} c_{\mathrm{apx}}^2 |||v_{\mathrm{CR}}|||_{\mathrm{NC}(\omega_z)}^2.$$

This, (4.2), and an overlapping argument show the local estimate

$$h_T^{-2}\|v_{\mathrm{CR}} - v_{\mathrm{C}}\|_{L^2(T)}^2 \le \frac{1}{4\sqrt{3}} \sum_{z\in\mathcal{N}(T)} e(z)^2 \le \sum_{z\in\mathcal{N}(T)} c_{\mathrm{apx}}^2 \frac{|||v_{\mathrm{CR}}|||_{\mathrm{NC}(\omega_z)}^2}{3} \le c_{\mathrm{apx}}^2 |||v_{\mathrm{CR}}|||_{\mathrm{NC}(\omega_T)}^2.$$

The sum over all $T \in \mathcal{T}$ and the previous arguments lead to

$$\|h_{\mathcal{T}}^{-1}(v_{\mathrm{CR}} - v_{\mathrm{C}})\|_{L^2(\Omega)}^2 \le \frac{c_{\mathrm{apx}}^2}{3} \sum_{z\in\mathcal{N}} |||v_{\mathrm{CR}}|||_{\mathrm{NC}(\omega_z)}^2 = c_{\mathrm{apx}}^2 |||v_{\mathrm{CR}}|||_{\mathrm{NC}}^2, \tag{4.5}$$

as desired. □

Examples 4.6. (1) One example of $J_C : CR^1_0(\mathcal{T}) \to S^1_0(\mathcal{T})$ with (4.1) is the enrichment operator $J_C := J_1$ (see [1, p. 297]) with

$$J_1 v_{CR}(z) := |\mathcal{T}(z)|^{-1} \sum_{T\in\mathcal{T}(z)} (v_{CR}|_T)(z) \quad \text{for any } z \in \mathcal{N}(\Omega). \tag{4.6}$$

(2) Another is the (possibly new) precise representation $J_C v_{CR} := I_C v^\star_{CR}$ with

$$I_C v^\star_{CR}(z) := (2\pi)^{-1} \sum_{T\in\mathcal{T}(z)} \measuredangle(T,z)(v_{CR}|_T)(z) \quad \text{for any } z \in \mathcal{N}(\Omega). \tag{4.7}$$

(3) Other examples are the maximum or minimum at each node,

$$J_C v_{CR}(z) := \max_{T\in\mathcal{T}(z)} (v_{CR}|_T)(z) \quad \text{for any } z \in \mathcal{N}(\Omega)$$

or

$$J_C v_{CR}(z) := \min_{T\in\mathcal{T}(z)} (v_{CR}|_T)(z) \quad \text{for any } z \in \mathcal{N}(\Omega).$$

(4) A discrete quasi-interpolation for the proof of optimal convergence rates of adaptive methods motivates the next example in a general formulation here. In the context of adaptive methods, $\mathcal{U} = \mathcal{T} \cap \hat{\mathcal{T}} \subseteq \mathcal{T}$ for a triangulation $\mathcal{T}$ and refinement $\hat{\mathcal{T}}$, see Remark 5.2. In a general setting, let $v_{CR} \in CR^1_0(\mathcal{T})$ and suppose there exists $\mathcal{U} \subseteq \mathcal{T}$ such that for any $K_1, K_2 \in \mathcal{U}$ with a shared node $z \in \mathcal{N}(K_1) \cap \mathcal{N}(K_2)$, the value of v_{CR} at z coincide, e.g., $v_{CR}|_{K_1}(z) = v_{CR}|_{K_2}(z)$. Hence, $J_{QI} v_{CR} \in S^1_0(\mathcal{T})$ is well-defined and satisfies (4.1) for

$$J_{QI} v_{CR}(z) := \begin{cases} v_{CR}|_K(z) & \text{if there exists } K \in \mathcal{U} \text{ with } z \in \mathcal{N}(K), \\ J_1 v_{CR}(z) & \text{else.} \end{cases} \tag{4.8}$$

Remark 4.7. Similar calculations with $2|e_{T_+}(z) - e_{T_-}(z)| \le \eta_E := |E| |[\frac{\partial v_{CR}}{\partial s}]_E|$ for $E \in \mathcal{E}(\Omega)$ in (4.3), $2|e_T(z)| \le \eta_E$ for $E \in \mathcal{E}(\partial\Omega)$, and $\sum_{E\in\mathcal{E}} \eta_E^2 \le 30 \cot(\omega_0) |||v_{CR} - v|||_{NC}$ for any $v \in H^1_0(\Omega)$ lead to a generalized version of Theorem 4.5 with $C_1^2 = 15 \cot(\omega_0)/(8\sqrt{3} \min\{1 - \cos(\pi/M_{int}), 1 - \cos(\pi/(M_{bd} + 1))\})$,

$$\|h_{\mathcal{T}}^{-1}(1 - J_C)v_{CR}\|_{L^2(\Omega)} \le C_1 \min_{v\in H^1_0(\Omega)} |||v_{CR} - v|||_{NC}.$$

Lemma 4.8. *For the special case $J_C = J_1$ from* (4.6)*, an improved constant in the estimate of Theorem 4.5 reads*

$$c_{apx}(J_1)^2 = \frac{\sqrt{3}\cot(\omega_0)}{2\min\{1 - \cos(\frac{2\pi}{M_{int}}), 1 - \cos(\frac{\pi}{M_{bd}})\}}.$$

Proof. The only change with respect to the proof of Theorem 4.5 concerns the estimate (4.4) of $e(z)^2$ for inner nodes $z \in \mathcal{N}(\Omega)$. Recall that for $z \in \mathcal{N}(\Omega)$ with patch $\mathcal{T}(z) = \{T_1, \dots, T_J\}$ and $e_j = v_{CR}|_{T_j}(z) - v_C(z)$ for $j = 1, \dots, J$, (4.3) shows

$$|e_j - e_{j+1}|^2 \le 2\cot(\omega_0) |||v_{CR}|||^2_{NC(T_j\cup T_{j+1})} \quad \text{for } j = 1, \dots, J$$

(with $e_{J+1} := e_0$ and $T_{J+1} := T_0$). Define $e = (e_1, \dots, e_J)^\top \in \mathbb{R}^J$ and

$$C = \begin{pmatrix} 2 & -1 & & & -1 \\ -1 & 2 & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & 2 & -1 \\ -1 & & & -1 & 2 \end{pmatrix} \in \mathbb{R}^{J\times J}.$$

Consequently,

$$e \cdot Ce = \sum_{j=1}^{J} |e_j - e_{j+1}|^2 \le 4\cot(\omega_0) |||v_{CR}|||^2_{NC(\omega_z)}. \tag{4.9}$$

For an approach similar to the one in the proof of Lemma 4.2, compute the eigenvalues $0 = \lambda_0 < \lambda_1 < \cdots < \lambda_K$ of the matrix $C \in \mathbb{R}^{J\times J}$ with $K := \lfloor \frac{J}{2} \rfloor$ with floor function $\lfloor\cdot\rfloor$ (i.e. $K = \frac{J}{2}$ for even J and $K = \frac{J-1}{2}$ for odd J), $\lambda_k = 2 - 2\cos(\frac{2k\pi}{J})$ (see [15, Theorem 3.4 (v)]) for $k = 0, \dots, \lfloor \frac{J}{2} \rfloor$. Indeed, the trigonometric addition formulas for sine and cosine show that the vectors $x^K, y^K \in \mathbb{R}^J$ with $x_j^k = \cos(\frac{2jk\pi}{J})$, $y_j^k = \sin(\frac{2jk\pi}{J})$ for $j = 1, \dots, J$, are the 0-vector or non-zero eigenvectors of C with eigenvalue λ_k for $k = 0, \dots, K$. An analysis of linear independence of $x^k, y^k \neq 0$ for even and odd J shows that there are J linearly independent eigenvectors. In any case, C is positive semi-definite with eigenvalues $0 = \lambda_0 < \lambda_1 < \cdots < \lambda_K$ and $\lambda_0 = 0$ is a simple eigenvalue with the eigenvector $u = (1, \dots, 1)^\top$ that is orthogonal to all other eigenvectors of C.

The identities $e = (v_{\mathrm{CR}}|_{T_1}(z), \dots, v_{\mathrm{CR}}|_{T_J}(z))^\top - v_{\mathrm{C}}(z)u$ and the definition of $v_{\mathrm{C}}(z)$ imply the orthogonality $e \cdot u = 0$. Hence, $\lambda_1 |e|^2 \le e \cdot Ce$ and therefore (4.9) shows

$$e(z)^2 = |e|^2 \le \frac{4\cot(\omega_0)}{\lambda_1} |||v_{\mathrm{CR}}|||_{\mathrm{NC}(\omega_z)} = \frac{2\cot(\omega_0)}{1 - \cos(\frac{2\pi}{J})} |||v_{\mathrm{CR}}|||_{\mathrm{NC}(\omega_z)}.$$

The remaining parts of the proof of Theorem 4.5 apply verbatim with different constants. □

Example 4.9. For the case of a triangulation of a convex domain with right isosceles triangles,

$$c_{\mathrm{apx}}(J_1) = \Big(\frac{\sqrt{3}}{2 - 2\cos(\frac{\pi}{4})}\Big)^{1/2} \le 1.6002.$$

The use of this discrete interpolation estimate enables a proof of the discrete Friedrichs inequality and an interpolation estimate for a new quasi-interpolation operator $J : H_0^1(\Omega) \to S_0^1(\mathcal{T})$ with the help of an inverse estimate.

Lemma 4.10 (Inverse Estimate). *Any $T \in \mathcal{T}$, $p_1 \in P_1(T)$, and the constant*

$$c_{\mathrm{inv}}^2 := 24\cot(\omega_0)(2\cot(\omega_0) - \cot(2\omega_0) + ((2\cot(\omega_0) - \cot(2\omega_0))^2 - 3)^{1/2})$$

satisfy

$$|||p_1|||_T \le c_{\mathrm{inv}} h_T^{-1} \|p_1\|_{L^2(T)}.$$

Proof. An analysis of the eigenvalues of the stiffness and the mass matrix and $\sigma = \sum_{z\in\mathcal{N}(T)} \cot(\measuredangle(T, z))$ leads to the local inverse estimate

$$|||p_1|||_T^2 \le \frac{6(\sigma + \sqrt{\sigma^2 - 3})\|p_1\|_{L^2(T)}^2}{|T|}.$$

A maximization shows $\sigma \le 2\cot(\omega_0) - \cot(2\omega_0)$ and $\frac{1}{|T|} \le h_T^{-2} 4\cot(\omega_0)$ concludes the proof. □

For right isosceles triangles, the constant $c_{\mathrm{inv}} = \sqrt{72}$ and all estimates in the proof are sharp.

Corollary 4.11 (Discrete Friedrichs Inequality). *Any $v_{\mathrm{CR}} \in \mathrm{CR}_0^1(\mathcal{T})$ and the constants*

$$c_{\mathrm{dF}} = h_{\max} c_{\mathrm{apx}}(J_1) + c_{\mathrm{F}}(\Omega)(1 + c_{\mathrm{inv}} c_{\mathrm{apx}}(J_1)) \quad \textit{and} \quad c_{\mathrm{F}}(\Omega) = \frac{\mathrm{width}(\Omega)}{\pi}$$

satisfy

$$\|v_{\mathrm{CR}}\|_{L^2(\Omega)} \le c_{\mathrm{dF}} |||v_{\mathrm{CR}}|||_{\mathrm{NC}}.$$

Proof. Given $v_{\mathrm{CR}} \in \mathrm{CR}_0^1(\mathcal{T})$, let $v_{\mathrm{C}} = J_1(v_{\mathrm{CR}})$ for the enrichment operator J_1 from Remark 4.6 so that Lemma 4.8 shows

$$\|v_{\mathrm{CR}} - v_{\mathrm{C}}\|_{L^2(\Omega)} \le h_{\max} c_{\mathrm{apx}}(J_1) |||v_{\mathrm{CR}}|||_{\mathrm{NC}}.$$

Lemma 4.10, the Friedrichs inequality $\|v_{\mathrm{C}}\|_{L^2(\Omega)} \le \frac{\mathrm{diam}(\Omega)|||v_{\mathrm{C}}|||}{\pi}$, and the triangle inequality yield

$$\begin{aligned} \|v_{\mathrm{C}}\|_{L^2(\Omega)} &\le c_{\mathrm{F}}(\Omega)|||v_{\mathrm{C}}|||_{\mathrm{NC}} \le c_{\mathrm{F}}(\Omega)(|||v_{\mathrm{CR}}||| + c_{\mathrm{inv}}\|h_{\mathcal{T}}^{-1}(v_{\mathrm{C}} - v_{\mathrm{CR}})\|_{L^2(\Omega)}) \\ &\le c_{\mathrm{F}}(\Omega)(1 + c_{\mathrm{inv}} c_{\mathrm{apx}}(J_1))|||v_{\mathrm{CR}}|||_{\mathrm{NC}}. \end{aligned} \tag{4.10}$$

The triangle inequality $\|v_{\mathrm{CR}}\|_{L^2(\Omega)} \le \|v_{\mathrm{CR}} - v_{\mathrm{C}}\|_{L^2(\Omega)} + \|v_{\mathrm{C}}\|_{L^2(\Omega)}$ concludes the proof. □

5 Quasi-Interpolation

This section proves an estimate for a quasi-interpolation operator $J : H^1_0(\Omega) \to S^1_0(\mathcal{T})$ as conclusion of the enrichment operator of Section 4. For $n = 2$, let $I_{\mathrm{NC}} : H^1_0(\Omega) \to \mathrm{CR}^1_0(\mathcal{T})$ denote the nonconforming interpolation operator with $(I_{\mathrm{NC}}v)(\mathrm{mid}(E)) = \fint_E v\,ds$ for all $E \in \mathcal{E}$ and $v \in H^1_0(\Omega)$.

Theorem 5.1 (Quasi-Interpolation). *The bounded linear projection $J := J_C \circ I_{\mathrm{NC}} : H^1_0(\Omega) \to S^1_0(\mathcal{T})$ for any mapping $J_{\mathrm{C}} : \mathrm{CR}^1_0(\mathcal{T}) \to S^1_0(\mathcal{T})$ with (4.1) and any $v \in H^1_0(\Omega)$ satisfy*

$$\|h_{\mathcal{T}}^{-1}(1-J)v\|_{L^2(\Omega)} \le (\kappa^2 + c_{\mathrm{apx}}^2)^{1/2} |||v|||,$$
$$|||Jv|||, |||(1-J)v||| \le c_{\mathrm{F}}(\Omega)(1 + c_{\mathrm{inv}} c_{\mathrm{apx}})|||v|||$$

with the constant $\kappa = (\frac{1}{48} + j_{1,1}^{-2})^{1/2}$ and the first positive root $j_{1,1}$ of the Bessel function of the first kind. Additionally, for any $T \in \mathcal{T}$, $f|_{\omega_T} \in S^1(\mathcal{T}(\omega_T))$ implies

$$f|_T = (Jf)|_T. \tag{5.1}$$

With

$$C_2 := \frac{\kappa+1}{j_{1,1}} + (1 + c_{\mathrm{inv}}) c_\omega c_{\mathrm{apx}} \Big(\frac{1}{j_{1,1}} + c(\mathcal{T})\Big),$$
$$c_\omega := \sin(\omega_0)^{-\max\{M_{\mathrm{bd}}-1, \frac{M_{\mathrm{int}}}{2}\}},$$
$$c(\mathcal{T}) := \max_{T\in\mathcal{T},\, z\in\mathcal{N}(T)} \left(\frac{\frac{1}{4} + \frac{2}{j_{1,1}^2}}{1 - |\cos(\measuredangle(T,z))|}\right)^{1/2},$$

any $v \in H^2(\Omega) \cap H^1_0(\Omega)$ additionally satisfies the second-order approximation property

$$\|h_{\mathcal{T}}^{-2}(1-J)v\|_{L^2(\Omega)} + \|h_{\mathcal{T}}^{-1}\nabla((1-J)v)\|_{L^2(\Omega)} \le C_2 \|\mathrm{D}^2 v\|_{L^2(\Omega)}.$$

Proof. For the proof of the first estimate, the triangle inequality implies

$$\|h_{\mathcal{T}}^{-1}(v - J_{\mathrm{C}} I_{\mathrm{NC}} v)\|_{L^2(\Omega)} \le \|h_{\mathcal{T}}^{-1}(v - I_{\mathrm{NC}} v)\|_{L^2(\Omega)} + \|h_{\mathcal{T}}^{-1}(1 - J_{\mathrm{C}}) I_{\mathrm{NC}} v)\|_{L^2(\Omega)}.$$

The interpolation estimate for the nonconforming interpolation operator with $\kappa = (\frac{1}{48} + j_{1,1}^{-2})^{1/2} = 0.29823$ (see [3]), Theorem 4.5, and the orthogonality of $\nabla_{\mathrm{NC}}(v - I_{\mathrm{NC}}v)$ onto $\nabla_{\mathrm{NC}} I_{\mathrm{NC}} v$ in $L^2(\Omega)$ yield

$$\|h_{\mathcal{T}}^{-1}(v - J_{\mathrm{C}} I_{\mathrm{NC}} v)\|_{L^2(\Omega)} \le (\kappa^2 + c_{\mathrm{apx}}^2)^{1/2} |||v|||.$$

For the second estimate, observe that $J : H^1_0(\Omega) \to H^1_0(\Omega)$ is a projection in $(H^1_0(\Omega), (\nabla\cdot, \nabla\cdot)_{L^2(\Omega)})$ and thus, $\|1 - J\|_{L(H^1_0(\Omega);H^1_0(\Omega))} = \|J\|_{L(H^1_0(\Omega);H^1_0(\Omega))}$ (see [11]). Consequently, (4.10) from the proof of the discrete Friedrichs inequality and $|||I_{\mathrm{NC}}v|||_{\mathrm{NC}} \le |||v|||$ show

$$\|J\|_{L(H^1_0(\Omega);H^1_0(\Omega))} \le c_{\mathrm{F}}(\Omega)(1 + c_{\mathrm{inv}} c_{\mathrm{apx}}).$$

For $T \in \mathcal{T}$ and $f|_{\omega_T} \in S^1(\mathcal{T}(\omega_T))$ as in (5.1), any $z \in \mathcal{N}(T)$ satisfies

$$(J_{\mathrm{C}}(I_{\mathrm{NC}} f))(z) = (J_{\mathrm{C}}(I_{\mathrm{NC}} f|_{\omega_z}))(z) = (J_{\mathrm{C}}(f|_{\omega_z}))(z) = f(z).$$

For the proof of the second-order approximation property, let $v \in H^2(\Omega) \cap H^1_0(\Omega)$ and let $Iv \in S^1_0(\mathcal{T})$ with $Iv(z) = v(z)$ be the nodal interpolant. Then $(1 - J_{\mathrm{C}})Iv = 0$ implies $(1 - J)v = (1 - I_{\mathrm{NC}})v + (1 - J_{\mathrm{C}})(I_{\mathrm{NC}} v - Iv)$. The triangle inequality yields

$$\|h_{\mathcal{T}}^{-2}(1-J)v\|_{L^2(\Omega)} \le \|h_{\mathcal{T}}^{-2}(1 - I_{\mathrm{NC}})v\|_{L^2(\Omega)} + \|h_{\mathcal{T}}^{-2}(1 - J_{\mathrm{C}})(I_{\mathrm{NC}} v - Iv)\|_{L^2(\Omega)}.$$

The second-order interpolation errors of nonconforming [3] and nodal interpolation [6] read

$$\|h_{\mathcal{T}}^{-2}(1 - I_{\mathrm{NC}})v\|_{L^2(\Omega)} \le \kappa \|h_{\mathcal{T}}^{-1}\nabla_{\mathrm{NC}}(1 - I_{\mathrm{NC}})v\|_{L^2(\Omega)} \le \frac{\kappa}{j_{1,1}} \|\mathrm{D}^2 v\|_{L^2(\Omega)},$$
$$\|h_{\mathcal{T}}^{-1}\nabla(1 - I)v\|_{L^2(\Omega)} \le c(\mathcal{T})\|\mathrm{D}^2 v\|_{L^2(\Omega)}.$$

Consequently, a slight modification of the proof of Theorem 4.5 in (4.5) with the estimate

$$h_T \le \max_{K\in\mathcal{T}(z)} h_K \le c_\omega h_T \quad \text{for any } z \in \mathcal{N},\ T \in \mathcal{T}(z),$$

and a triangle inequality imply

$$\begin{aligned}\|h_{\mathcal{T}}^{-2}(1-J_{\mathrm{C}})(I_{\mathrm{NC}}v - Iv)\|_{L^2(\Omega)} &\le c_\omega c_{\mathrm{apx}} \|h_{\mathcal{T}}^{-1}\nabla_{\mathrm{NC}}(I_{\mathrm{NC}}v - Iv)\|_{L^2(\Omega)}\\ &\le c_\omega c_{\mathrm{apx}}(j_{1,1}^{-1} + c(\mathcal{T}))\|\mathrm{D}^2 v\|_{L^2(\Omega)}.\end{aligned}$$

This results in the estimate of the first term in the assertion

$$\|h_{\mathcal{T}}^{-2}(1-J)v\|_{L^2(\Omega)} \le (\kappa/j_{1,1} + c_\omega c_{\mathrm{apx}}(j_{1,1}^{-1} + c(\mathcal{T})))\|\mathrm{D}^2 v\|_{L^2(\Omega)}.$$

The split from above yields

$$\|h_{\mathcal{T}}^{-1}\nabla((1-J)v)\|_{L^2(\Omega)} \le \|h_{\mathcal{T}}^{-1}\nabla((1-I_{\mathrm{NC}})v)\|_{L^2(\Omega)} + \|h_{\mathcal{T}}^{-1}\nabla((1-J_{\mathrm{C}})(I_{\mathrm{NC}}v - Iv))\|_{L^2(\Omega)}.$$

The inverse estimate leads to $\|h_{\mathcal{T}}^{-1}\nabla((1-J_{\mathrm{C}})(I_{\mathrm{NC}}v - Iv))\|_{L^2(\Omega)} \le c_{\mathrm{inv}}\|h_{\mathcal{T}}^{-2}(1-J_{\mathrm{C}})(I_{\mathrm{NC}}v - Iv)\|_{L^2(\Omega)}$ and therefore

$$\|h_{\mathcal{T}}^{-1}\nabla((1-J)v)\|_{L^2(\Omega)} \le (j_{1,1}^{-1} + c_{\mathrm{inv}} c_\omega c_{\mathrm{apx}}(j_{1,1}^{-1} + c(\mathcal{T})))\|\mathrm{D}^2 v\|_{L^2(\Omega)}.$$ □

Remark 5.2 (Discrete Quasi-Interpolation). Consider a triangulation $\mathcal{T}$ and refinement $\hat{\mathcal{T}}$. For any $\hat{v}_C \in S_0^1(\hat{\mathcal{T}})$ and $K \in \mathcal{U} := \mathcal{T} \cap \hat{\mathcal{T}}$, $I_{\mathrm{NC}}\hat{v}_C|_K = \hat{v}_C|_K$. Hence, any $K_1, K_2 \in \mathcal{U}$ with $z \in \mathcal{N}(K_1) \cap \mathcal{N}(K_2)$ satisfy

$$I_{\mathrm{NC}}\hat{v}_C|_{K_1}(z) = \hat{v}_C(z) = I_{\mathrm{NC}}\hat{v}_C|_{K_2}(z).$$

Consequently, the application of Theorem 5.1 with $J_{\mathrm{C}} = J_{QI}$ from (4.8) yields a discrete quasi-interpolation $J_{\mathrm{dQI}} := J_{QI} \circ I_{\mathrm{NC}}|_{S_0^1(\hat{\mathcal{T}})} : S_0^1(\hat{\mathcal{T}}) \to S_0^1(\mathcal{T})$ such that any $\hat{v}_C \in S_0^1(\hat{\mathcal{T}})$ satisfies $\hat{v}_C = J_{\mathrm{dQI}}\hat{v}_C$ on $\mathcal{T} \cap \hat{\mathcal{T}}$ and

$$\|h_{\mathcal{T}}^{-1}(1-J_{\mathrm{dQI}})\hat{v}_C\|_{L^2(\Omega)} \le (\kappa^2 + c_{\mathrm{apx}}^2)^{1/2}|||\hat{v}_C|||. \tag{5.2}$$

A thorough inspection of the proofs of Theorems 4.5 and 5.1 shows that this interpolation operator can be extended to $J_{\mathrm{dQI}} : S^1(\hat{\mathcal{T}}) \to S^1(\mathcal{T})$ with the same properties and constant

$$c_{\mathrm{apx}}^2 = \frac{\sqrt{3}\cot(\omega_0)}{2\min\{1-\cos(\frac{\pi}{M_{\mathrm{int}}}), 1-\cos(\frac{\pi}{2M_{\mathrm{bd}}-1})\}}$$

arising from the eigenvalue problem [15, Theorem 3.2 (viii)].

6 Constants in the Axioms of Adaptivity

This section recapitulates the proof of optimal convergence rates of the Courant and the Crouzeix–Raviart FEM in two dimensions in the axiomatic framework of [2, 8] with explicit constants. Define $a(u, v) := (\nabla u, \nabla v)_{L^2(\Omega)}$ for any $v, w \in H_0^1(\Omega)$. Given $f \in L^2(\Omega)$, the CFEM seeks $u_C \in S_0^1(\mathcal{T})$ with

$$a(u_C, v_C) = (f, v_C)_{L^2(\Omega)} \quad \text{for any } v_C \in S_0^1(\mathcal{T}). \tag{6.1}$$

For any admissible triangulation $\mathcal{T} \in \mathbb{T}$ with CFEM solution $u_C \in \mathrm{CR}_0^1(\mathcal{T})$ to (6.1) and $K \in \mathcal{T}$, define

$$\eta_C^2(\mathcal{T}, K) := |K|\|f\|_{L^2(K)}^2 + |K|^{1/2} \sum_{E\in\mathcal{E}(K)\cap\mathcal{E}(\Omega)} \|[\nabla u_C \cdot \nu_E]\|_{L^2(E)}^2.$$

For $\mathcal{T} \in \mathbb{T}$ and refinement $\hat{\mathcal{T}}$ with solutions $u_C \in S_0^1(\mathcal{T})$ and $\hat{u}_C \in S_0^1(\hat{\mathcal{T}})$, define

$$\delta_C(\mathcal{T}, \hat{\mathcal{T}}) := |||u_C - \hat{u}_C|||.$$

The optimality proof of [2] relies on axioms (A1)–(A4) below with constants $0 < \Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4 < \infty$ and $0 < \varrho_2 < 1$. Any $\mathcal{T} \in \mathbb{T}$ and refinement $\hat{\mathcal{T}}$ satisfy Stability (A1)

$$|\eta_C(\mathcal{T}, \mathcal{T} \cap \hat{\mathcal{T}}) - \eta_C(\hat{\mathcal{T}}, \mathcal{T} \cap \hat{\mathcal{T}})| \le \Lambda_1 \delta_C(\mathcal{T}, \hat{\mathcal{T}}) \tag{6.2}$$

and Reduction (A2)

$$\eta_C(\hat{\mathcal{T}}, \hat{\mathcal{T}} \setminus \mathcal{T}) \leq \varrho_2 \eta_C(\mathcal{T}, \mathcal{T} \setminus \hat{\mathcal{T}}) + \Lambda_2 \delta_C(\mathcal{T}, \hat{\mathcal{T}}).$$

Moreover, [2] shows discrete reliability (A3) on a simply-connected domain $\Omega \subseteq \mathbb{R}^2$,

$$\delta_C^2(\mathcal{T}, \hat{\mathcal{T}}) \leq \Lambda_3 \eta_C^2(\mathcal{T}, \mathcal{T} \setminus \hat{\mathcal{T}}). \tag{6.3}$$

The Quasi-Orthogonality (A4) shows that the output $\mathcal{T}_k$, $k = 1, 2, \dots$, of the adaptive algorithm with corresponding quantities $\eta_k := \eta_C(\mathcal{T}_k, \mathcal{T}_k)$ and any $\ell, m \in \mathbb{N}$ satisfy

$$\sum_{k=\ell}^{\ell+m} \delta_C^2(\mathcal{T}_k, \mathcal{T}_{k+1}) \leq \Lambda_4 \eta_\ell^2.$$

The main result [2, Theorem 4.5] and the axioms of adaptivity state that (A1)–(A4) with the above-mentioned constants yield optimal convergence rates of the adaptive Crouzeix–Raviart FEM with Dörfler marking for any bulk parameter

$$0 < \theta < \theta_0 := (1 + \Lambda_1^2 \Lambda_3)^{-1}. \tag{6.4}$$

This is a sufficient condition for optimal rates and requires the quantification of θ_0 and so to calculate Λ_1 and Λ_3 explicitly.

The proof of Stability (A1) is essentially contained in [9] but is included here for explicit gathering of the constants.

Theorem 6.1 (Stability (A1) for CFEM). *The constants*

$$c_{\mathrm{quot}} := \max_{K_1, K_2 \in \mathcal{T},\ \mathcal{E}(K_1) \cap \mathcal{E}(K_2) \neq \emptyset} \frac{|K_1|}{|K_2|} \leq \frac{2 \cot(\omega_0)}{\sin(\omega_0)}$$

and

$$\Lambda_1^2 = 6 \cot(\omega_0)^{1/2} (1 + c_{\mathrm{quot}}^{1/2})^2$$

satisfy (6.2).

Proof. The reverse triangle inequality for vectors with entries $|T|^{1/4} \|[\nabla u_C \cdot \nu_E]_E\|_{L^2(E)}$ resp. $|T|^{1/4} \|[\frac{\partial \hat{u}_C}{\partial s}]_E\|_{L^2(E)}$ for any $T \in \mathcal{T} \cap \hat{\mathcal{T}}$ and $E \in \mathcal{E}(T)$ shows

$$|\eta_C(\mathcal{T}, \mathcal{T} \cap \hat{\mathcal{T}}) - \eta_C(\hat{\mathcal{T}}, \mathcal{T} \cap \hat{\mathcal{T}})|^2 \leq \sum_{T \in \mathcal{T} \cap \hat{\mathcal{T}}} \sum_{E \in \mathcal{E}(T)} |T|^{1/2} (\|[\nabla u_C \cdot \nu_E]_E\|_{L^2(E)} - \|[\nabla \hat{u}_C \cdot \nu_E]_E\|_{L^2(E)})^2.$$

Furthermore, the reverse triangle inequality in $L^2(E)$ imply that any $T \in \mathcal{T} \cap \hat{\mathcal{T}}$ and $E \in \mathcal{E}(T)$ satisfy

$$\big| \|[\nabla u_C \cdot \nu_E]_E\|_{L^2(E)} - \|[\nabla \hat{u}_C \cdot \nu_E]_E\|_{L^2(E)} \big| \leq \|[\nabla_{\mathrm{NC}}(u_C - \hat{u}_C)]_E\|_{L^2(E)}.$$

The triangle inequality and the trace identity show that the function $\hat{p}_0 := \nabla_{\mathrm{NC}}(u_C - \hat{u}_C) \in P_0(\hat{\mathcal{T}}; \mathbb{R}^2)$ satisfies on $\partial T_+ \cap \partial T_- = E \in \hat{\mathcal{E}}(\Omega)$ with $\hat{T}_+, \hat{T}_- \in \hat{\mathcal{T}}$,

$$\begin{aligned} \|[\hat{p}_0]_E\|_{L^2(E)}^2 &\leq (\|\hat{p}_0|_{T_+}\|_{L^2(E)} + \|\hat{p}_0|_{T_-}\|_{L^2(E)})^2 \\ &= |E| (|\hat{T}_+|^{-1/2} \|\hat{p}_0\|_{L^2(T_+)} + |\hat{T}_-|^{-1/2} \|\hat{p}_0\|_{L^2(T_-)})^2 \\ &\leq |E| (|\hat{T}_+|^{-1} + |\hat{T}_-|^{-1}) \|\hat{p}_0\|_{L^2(\hat{\omega}_E)}^2. \end{aligned}$$

The estimates $|\hat{T}_+|^{1/2} + |\hat{T}_-|^{1/2} \leq |\hat{T}_-|^{1/2}(1 + c_{\mathrm{quot}}^{1/2})$ and $|\hat{T}_\pm|^{-1/2} \leq 2 \cot(\omega_0)^{1/2} |E|^{-1}$ show

$$\begin{aligned} (|\hat{T}_+|^{1/2} + |\hat{T}_-|^{1/2}) |E| (|\hat{T}_-|^{-1} + |\hat{T}_+|^{-1}) &\leq |E| (1 + c_{\mathrm{quot}}^{1/2}) (|\hat{T}_-|^{-1/2} + |\hat{T}_+|^{-1} |\hat{T}_-|^{1/2}) \\ &\leq 2 \cot(\omega_0)^{1/2} (1 + c_{\mathrm{quot}}^{1/2}) (1 + |\hat{T}_+|^{-1/2} |\hat{T}_-|^{1/2}) \\ &\leq 2 \cot(\omega_0)^{1/2} (1 + c_{\mathrm{quot}}^{1/2})^2 =: c_{\mathrm{sr}}. \end{aligned}$$

The estimates $|\hat{T}_\pm|^{-1} \leq 4 \cot(\omega_0) |E|^{-2}$, $|\hat{T}_\pm| \leq \frac{1}{2} |E| h_{\hat{T}_\pm}$, and $h_{\hat{T}_\pm} \leq \frac{|E|}{\sin(\omega_0)}$ imply

$$c_{\mathrm{quot}} \leq \frac{2 \cot(\omega_0)}{\sin(\omega_0)}.$$

The summation over $\mathcal{T}\cap\hat{\mathcal{T}}$ and the finite overlap of $(\hat{\omega}_E)_{E\in\hat{\mathcal{E}}}$ leads to

$$|\eta_C(\mathcal{T},\mathcal{T}\cap\hat{\mathcal{T}})-\eta_C(\hat{\mathcal{T}},\mathcal{T}\cap\hat{\mathcal{T}})|^2 \le c_{\mathrm{sr}}\sum_{E\in\hat{\mathcal{E}}}\|\nabla_{\mathrm{NC}}(u_C-\hat{u}_C)\|^2_{L^2(\hat{\omega}_E)} \le 3c_{\mathrm{sr}}\|\nabla_{\mathrm{NC}}(u_C-\hat{u}_C)\|^2_{L^2(\Omega)}. \qquad \square$$

Theorem 6.2 (Discrete Reliability (A3) for CFEM). *The constant*

$$\Lambda_3 = 4\cot(\omega_0)(\kappa^2+c_{\mathrm{apx}}^2)(1+6\cot(\omega_0)^{1/2}(1+c_{\mathrm{inv}}))$$

satisfies (6.3).

Proof. With solution $u_C\in S^1_0(\mathcal{T})$ (resp. $\hat{u}_C\in S^1_0(\hat{\mathcal{T}})$) to the discrete problem with respect to $\mathcal{T}\in\mathbb{T}$ (resp. $\hat{\mathcal{T}}\in\mathbb{T}(\mathcal{T})$), define $\hat{e}_C := \hat{u}_C - u_C$ and discrete quasi-interpolation $e_C\in S^1(\mathcal{T})$ of $\hat{e}_C\in S^1(\hat{\mathcal{T}})$ from Remark 5.2. The Galerkin orthogonality $a(\hat{e}_C, e_C)=0$, $\hat{e}_C-e_C=0$ on $\mathcal{T}\cap\hat{\mathcal{T}}$ and a piecewise integration by parts show

$$\begin{aligned}\delta_C^2(\mathcal{T},\hat{\mathcal{T}}) &= a(\hat{u}_C,\hat{e}_C-e_C)-a(u_C,\hat{e}_C-e_C)\\ &= \int_{\mathcal{T}\setminus\hat{\mathcal{T}}}(h_{\mathcal{T}}f)h_{\mathcal{T}}^{-1}(\hat{e}_C-e_C)\,dx - \sum_{E\in\mathcal{E}(\Omega)\cap\mathcal{E}(\mathcal{T}\setminus\mathcal{T})}\int_E[\nabla u_C\cdot\nu_E](\hat{e}_C-e_C)\,ds.\end{aligned}$$

The Cauchy and the trace inequality (6.8) prove

$$\delta_C^2(\mathcal{T},\hat{\mathcal{T}}) \le \Big(\|h_{\mathcal{T}}f\|_{L^2(\mathcal{T}\setminus\hat{\mathcal{T}})}+\sqrt{3}c_{\mathrm{tr}}\sqrt{\sum_{E\in\mathcal{E}(\mathcal{T}\setminus\hat{\mathcal{T}})}|E|\|[\nabla u_C\cdot\nu_E]_E\|^2_{L^2(E)}}\Big)\|h_{\mathcal{T}}^{-1}(\hat{e}_C-e_C)\|_{L^2(\Omega)}.$$

The estimates $h_K^2\le 4\cot(\omega_0)|K|$, $|E|\le 2\cot(\omega_0)^{1/2}|K|^{1/2}$ for any $K\in\mathcal{T}$ and the first-order approximation property (5.2) prove the assertion with $\Lambda_3=(\kappa^2+c_{\mathrm{apx}}^2)(4\cot(\omega_0)+6c_{\mathrm{tr}}^2\cot(\omega_0)^{1/2})$. $\square$

Example 6.3. For right isosceles triangles, $\Lambda_1^2\le 40.36$, $\Lambda_3\le 9201$ and (6.4) lead to $\theta_0\ge 2.6\times10^{-6}$ for the Courant FEM, despite the general wisdom that $\theta=0.3$ leads to optimal convergence.

The remaining part of this section proves an explicit bound for the bulk parameter for the Crouzeix–Raviart FEM with solution $u_{\mathrm{CR}}\in \mathrm{CR}^1_0(\mathcal{T})$ to $a_{\mathrm{NC}}(u_{\mathrm{CR}},v_{\mathrm{CR}})=(f,v_{\mathrm{CR}})_{L^2(\Omega)}$ for any $v_{\mathrm{CR}}\in\mathrm{CR}^1_0(\mathcal{T})$ with

$$a_{\mathrm{NC}}(v_{\mathrm{CR}},w_{\mathrm{CR}}) := (\nabla_{\mathrm{NC}}v_{\mathrm{CR}},\nabla_{\mathrm{NC}}w_{\mathrm{CR}})_{L^2(\Omega)}.$$

For any admissible triangulation $\mathcal{T}\in\mathbb{T}$ and $K\in\mathcal{T}$, define

$$\eta^2_{\mathrm{CR}}(\mathcal{T},K) := |K|\|f\|^2_{L^2(K)}+|K|^{1/2}\sum_{E\in\mathcal{E}(K)}\Big\|\Big[\frac{\partial u_{\mathrm{CR}}}{\partial s}\Big]\Big\|^2_{L^2(E)}.$$

For $\mathcal{T}\in\mathbb{T}$ and refinement $\hat{\mathcal{T}}$ with solutions $u_{\mathrm{CR}}\in\mathrm{CR}^1_0(\mathcal{T})$ and $\hat{u}_{\mathrm{CR}}\in\mathrm{CR}^1_0(\hat{\mathcal{T}})$, define

$$\delta_{\mathrm{CR}}(\mathcal{T},\hat{\mathcal{T}}) := |||u_{\mathrm{CR}}-\hat{u}_{\mathrm{CR}}|||_{\mathrm{NC}}.$$

The proof of Stability (A1) from Theorem 6.4 applies verbatim with $\frac{\partial}{\partial\nu_E}$ replaced by τ_E in $\partial/\partial s$.

Theorem 6.4 (Stability (A1) for CRFEM). *The constants c_{quot} from Theorem 6.1 and*

$$\Lambda_1^2 = 48\cot(\omega_0)(2\sin(\omega_0))^{-1/2}$$

satisfy (6.2).

Theorem 6.5 (Discrete Reliability (A3) for CRFEM). *For a simply-connected domain $\Omega\subset\mathbb{R}^2$, the constant*

$$\Lambda_3 = 12\cot(\omega_0)(\kappa^2+c_{\mathrm{apx}}^2)(1+c_{\mathrm{inv}})$$

satisfies (6.3).

Proof. Given the solution $u_{\mathrm{CR}}\in\mathrm{CR}^1_0(\mathcal{T})$ (resp. $\hat{u}_{\mathrm{CR}}\in\mathrm{CR}^1_0(\hat{\mathcal{T}})$) to the discrete problem with respect to $\mathcal{T}\in\mathbb{T}$ (resp. $\hat{\mathcal{T}}\in\mathbb{T}(\mathcal{T})$), consider a discrete Helmholtz decomposition of $\nabla_{\mathrm{NC}}u_{\mathrm{CR}}\in P_0(\mathcal{T};\mathbb{R}^2)\subseteq P_0(\hat{\mathcal{T}};\mathbb{R}^2)$,

$$\nabla_{\mathrm{NC}}u_{\mathrm{CR}} = \nabla_{\mathrm{NC}}\hat{\alpha}_{\mathrm{CR}}+\operatorname{Curl}\hat{\beta}_C \tag{6.5}$$

for unique $\hat{\alpha}_{\mathrm{CR}} \in \mathrm{CR}^1_0(\hat{\mathcal{T}})$ and $\hat{\beta}_C \in S^1(\hat{\mathcal{T}})/\mathbb{R}$ so that

$$\delta^2_{\mathrm{CR}}(\mathcal{T}, \hat{\mathcal{T}}) = |||u_{\mathrm{CR}} - \hat{u}_{\mathrm{CR}}|||^2_{\mathrm{NC}} = |||\hat{\alpha}_{\mathrm{CR}} - \hat{u}_{\mathrm{CR}}|||^2_{\mathrm{NC}} + |||\hat{\beta}_C|||^2. \tag{6.6}$$

Abbreviate $\hat{v}_{\mathrm{CR}} := \hat{u}_{\mathrm{CR}} - \hat{\alpha}_{\mathrm{CR}} \in \mathrm{CR}^1_0(\hat{\mathcal{T}})$ and $v_{\mathrm{CR}} := I_{\mathrm{NC}}\hat{v}_{\mathrm{CR}} \in \mathrm{CR}^1_0(\mathcal{T})$. An analogous proof to the interpolation estimate for $I_{\mathrm{NC}} : H^1_0(\Omega) \to H^1_0(\Omega)$ (see [5, Theorem 2.1]) with the discrete Poincaré constant $c_{\mathrm{P}} = \sqrt{\frac{3}{8}}$ from Theorem 3.1 and the discrete trace identity (Lemma 2.6) yields $\kappa_{\mathrm{CR}} := (\frac{1}{8} + c^2_{\mathrm{P}})^{1/2} = 2^{-1/2}$ with

$$\|h^{-1}_{\mathcal{T}}(\hat{v}_{\mathrm{CR}} - v_{\mathrm{CR}})\|_{L^2(\Omega)} \le \kappa_{\mathrm{CR}}|||\hat{v}_{\mathrm{CR}}|||_{\mathrm{NC}}.$$

Since $\hat{u}_{\mathrm{CR}}$ solves the discrete problem on $\hat{\mathcal{T}}$,

$$|||\hat{u}_{\mathrm{CR}} - \hat{\alpha}_{\mathrm{CR}}|||^2_{\mathrm{NC}} = a_{\mathrm{NC}}(\hat{u}_{\mathrm{CR}}, \hat{v}_{\mathrm{CR}}) - a_{\mathrm{NC}}(\hat{\alpha}_{\mathrm{CR}}, \hat{v}_{\mathrm{CR}}) = F(\hat{v}_{\mathrm{CR}}) - a_{\mathrm{NC}}(\hat{\alpha}_{\mathrm{CR}}, \hat{v}_{\mathrm{CR}}).$$

The orthogonal decomposition (6.5) and $\Pi_0\nabla_{\mathrm{NC}}\hat{v}_{\mathrm{CR}} = \nabla_{\mathrm{NC}}I_{\mathrm{NC}}\hat{v}_{\mathrm{CR}} = \nabla_{\mathrm{NC}}v_{\mathrm{CR}}$ imply

$$a_{\mathrm{NC}}(\hat{\alpha}_{\mathrm{CR}}, \hat{v}_{\mathrm{CR}}) = (\nabla_{\mathrm{NC}}u_{\mathrm{CR}}, \nabla_{\mathrm{NC}}\hat{v}_{\mathrm{CR}}) = (\nabla_{\mathrm{NC}}u_{\mathrm{CR}}, \nabla_{\mathrm{NC}}v_{\mathrm{CR}}) = F(v_{\mathrm{CR}}).$$

The three last displayed formulas, the Cauchy inequality and $\hat{v}_{\mathrm{CR}} - v_{\mathrm{CR}} = 0$ on $\mathcal{T} \cap \hat{\mathcal{T}}$ yield

$$|||\hat{u}_{\mathrm{CR}} - \hat{\alpha}_{\mathrm{CR}}|||^2_{\mathrm{NC}} = F(\hat{v}_{\mathrm{CR}} - v_{\mathrm{CR}}) = (f, \hat{v}_{\mathrm{CR}} - v_{\mathrm{CR}})_{L^2(\mathcal{T}\setminus\hat{\mathcal{T}})} \le \kappa_{\mathrm{CR}}\|h_{\mathcal{T}}f\|_{L^2(\mathcal{T}\setminus\hat{\mathcal{T}})}|||\hat{u}_{\mathrm{CR}} - \hat{\alpha}_{\mathrm{CR}}|||_{\mathrm{NC}}.$$

This and $h^2_K \le 4\cot(\omega_0)|K|$ for $K \in \mathcal{T}$ show

$$2|||\hat{u}_{\mathrm{CR}} - \hat{\alpha}_{\mathrm{CR}}|||^2_{\mathrm{NC}} \le \|h_{\mathcal{T}}f\|^2_{L^2(\mathcal{T}\setminus\hat{\mathcal{T}})} \le 4\cot(\omega_0) \sum_{K\in\mathcal{T}\setminus\hat{\mathcal{T}}} |K|\|f\|^2_{L^2(K)}. \tag{6.7}$$

The estimate of $|||\hat{\beta}_C|||$ utilizes the discrete quasi-interpolation $\beta_C \in S^1(\mathcal{T})$ of $\hat{\beta}_C \in S^1(\hat{\mathcal{T}})$ from Remark 5.2. A piecewise integration by parts, $\hat{\beta}_C = \beta_C$ on $\mathcal{T} \cap \hat{\mathcal{T}}$, and $\mathcal{E}(\mathcal{T} \setminus \hat{\mathcal{T}}) := \bigcup_{K\in\mathcal{T}\setminus\hat{\mathcal{T}}} \mathcal{E}(K)$ show

$$\begin{aligned} |||\hat{\beta}_C|||^2 &= \int_\Omega \operatorname{Curl}\hat{\beta}_C \cdot \nabla_{\mathrm{NC}}u_{\mathrm{CR}}\,\mathrm{d}x = \int_\Omega \operatorname{Curl}(\hat{\beta}_C - \beta_C) \cdot \nabla_{\mathrm{NC}}u_{\mathrm{CR}}\,\mathrm{d}x \\ &= \sum_{K\in\mathcal{T}\setminus\hat{\mathcal{T}}} \int_K \operatorname{Curl}(\hat{\beta}_C - \beta_C) \cdot \nabla_{\mathrm{NC}}u_{\mathrm{CR}}\,\mathrm{d}x = \sum_{K\in\mathcal{T}\setminus\hat{\mathcal{T}}} \int_{\partial K} (\hat{\beta}_C - \beta_C)\frac{\partial u_{\mathrm{CR}}}{\partial s}\,\mathrm{d}s \\ &= \sum_{E\in\mathcal{E}(\mathcal{T}\setminus\hat{\mathcal{T}})} \int_E (\hat{\beta}_C - \beta_C)\Big[\frac{\partial u_{\mathrm{CR}}}{\partial s}\Big]_E\,\mathrm{d}s. \end{aligned}$$

The trace identity on any $T \in \mathcal{T}$ and $E \in \mathcal{E}(T)$ with $v := (\hat{\beta}_C - \beta_C)^2$ and the Cauchy inequality lead to

$$|E|^{-1}\|\hat{\beta}_C - \beta_C\|^2_{L^2(E)} \le |T|^{-1}(\|\hat{\beta}_C - \beta_C\|^2_{L^2(T)} + h_T\|\hat{\beta}_C - \beta_C\|_{L^2(T)}|||\hat{\beta}_C - \beta_C|||_{\mathrm{NC}(T)}).$$

The estimate $|T|^{-1} \le 4\cot(\omega_0)h^{-2}_T$ and the weighted Young inequality for any $\lambda > 0$ show

$$|E|^{-1}\|\hat{\beta}_C - \beta_C\|^2_{L^2(E)} \le 4\cot(\omega_0)((1 + (2\lambda)^{-1})\|h^{-1}_{\mathcal{T}}(\hat{\beta}_C - \beta_C)\|^2_{L^2(T)} + \lambda/2|||\hat{\beta}_C - \beta_C|||^2_{\mathrm{NC}(T)}).$$

Hence, the inverse estimate and the direct minimization $\min_{\lambda>0}((2\lambda)^{-1} + \frac{c^2_{\mathrm{inv}}\lambda}{2}) = c_{\mathrm{inv}}$ prove, for the constant $c^2_{\mathrm{tr}} := 4\cot(\omega_0)(1 + c_{\mathrm{inv}})$, the trace inequality

$$|E|^{-1}\|\hat{\beta}_C - \beta_C\|^2_{L^2(E)} \le c^2_{\mathrm{tr}}\|h^{-1}_{\mathcal{T}}(\hat{\beta}_C - \beta_C)\|^2_{L^2(\omega_E)}. \tag{6.8}$$

This and the Cauchy inequality imply

$$\begin{aligned} |||\hat{\beta}_C|||^2 &\le \sum_{E\in\mathcal{E}(\mathcal{T}\setminus\hat{\mathcal{T}})} \int_E |E|^{-1/2}|\hat{\beta}_C - \beta_C||E|^{1/2}\Big|\Big[\frac{\partial u_{\mathrm{CR}}}{\partial s}\Big]_E\Big|\,\mathrm{d}s \\ &\le \sqrt{\sum_{E\in\mathcal{E}(\mathcal{T}\setminus\hat{\mathcal{T}})} |E|^{-1}\|\hat{\beta}_C - \beta_C\|^2_{L^2(E)}}\sqrt{\sum_{E\in\mathcal{E}(\mathcal{T}\setminus\hat{\mathcal{T}})} |E|\Big\|\Big[\frac{\partial u_{\mathrm{CR}}}{\partial s}\Big]_E\Big\|^2_{L^2(E)}} \\ &\le \sqrt{3}c_{\mathrm{tr}}\|h^{-1}_{\mathcal{T}}(\hat{\beta}_C - \beta_C)\|_{L^2(\Omega)}\sqrt{\sum_{E\in\mathcal{E}(\mathcal{T}\setminus\hat{\mathcal{T}})} |E|\Big\|\Big[\frac{\partial u_{\mathrm{CR}}}{\partial s}\Big]_E\Big\|^2_{L^2(E)}}. \end{aligned}$$

The first-order approximation property (5.2) of the discrete quasi-interpolation, $|E| \le 2\cot(\omega_0)^{1/2}|T|^{1/2}$, (6.6) and (6.7) with $2\cot(\omega_0) \le 24\cot(\omega_0)^{3/2}(\kappa^2 + c^2_{\mathrm{apx}})(1 + c_{\mathrm{inv}})$ conclude the proof. □

Example 6.6. For right isosceles triangles, it holds

$$\Lambda_1^2 \le 34.97 \quad \text{and} \quad \Lambda_3 \le 4521$$

and (6.4) leads to $\theta_0 \ge 6.3 \times 10^{-6}$ for the Crouzeix–Raviart FEM, despite the general wisdom that $\theta = 0.3$ leads to optimal convergence.

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