CPDE II, SuSe 19
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Exercise Sheet 1

Discussion on 30 April 2019

Please be prepared to present one or more of the following exercises on the blackboard. If not stated otherwise, algorithms or code can be displayed in Matlab or pseudocode.

Exercise 1 (Normal trace operator). Let $\Omega \subset \mathbb{R}^d$ be a polyhedral domain whose boundary $\partial\Omega$ is locally the graph of a Lipschitz function and $\nu : \partial\Omega \to \mathbb{R}^d$ be the outward unit normal vector field. Recall that there exists a bounded linear trace operator $\gamma_0 : H^1(\Omega) \to L^2(\partial\Omega)$ with

$$\gamma_0(\varphi) = \varphi|_{\partial\Omega} \quad \text{for all} \quad \varphi \in H^1(\Omega) \cap C(\overline{\Omega})$$

and

$$\|\gamma_0(v)\|_{L^2(\partial\Omega)} \leq C\|v\|_{H^1(\Omega)} \quad \text{for all} \quad v \in H^1(\Omega).$$

Define the trace space $H^{1/2}(\partial\Omega) := \gamma_0(H^1(\Omega))$ equipped with the minimal extension norm

$$\|g\|_{H^{1/2}(\partial\Omega)} := \inf\left\{\|w\|_{H^1(\Omega)} \ : \ w \in H^1(\Omega) \text{ with } \gamma_0(w) = g\right\}.$$

(a) Show that γ_0 remains bounded with respect to the $H^{1/2}(\partial\Omega)$ norm.

(b) Let $H^{-1/2}(\partial\Omega) := (H^{1/2}(\partial\Omega))^*$ be the dual space. Use Green's theorem to define an operator $\gamma_\nu : H(\mathrm{div}, \Omega) \to H^{-1/2}(\partial\Omega)$ with

$$\gamma_\nu(\varphi) = \varphi \cdot \nu|_{\partial\Omega} \quad \text{for all} \quad \varphi \in H(\mathrm{div}, \Omega) \cap C(\overline{\Omega}; \mathbb{R}^d).$$

Thus, given any $q \in H(\mathrm{div}, \Omega)$, define a dual object $\gamma_\nu(q) \in H^{-1/2}(\partial\Omega)$ determining the values of $\langle \gamma_\nu(q), \gamma_0(v)\rangle_{\partial\Omega}$ for every $v \in H^1(\Omega)$.

(c) Prove that γ_ν is a bounded operator with respect to the dual norm

$$\|g^*\|_{H^{-1/2}(\partial\Omega)} := \sup\left\{\langle g^*, g\rangle_{\partial\Omega} \ : \ g \in H^{1/2}(\Omega),\ \|g\|_{H^{1/2}(\Omega)} = 1\right\}.$$

Exercise 2 (Piola transformation). Let $T_{\mathrm{ref}} := \mathrm{conv}\{(1,0,0)^\top, (0,1,0)^\top, (0,0,1)^\top, (0,0,0)^\top\}$ denote the reference simplex and $T = \mathrm{conv}\{P_1, P_2, P_3, P_4\}$ an arbitrary non-degenerated simplex.

(a) Sketch T_{ref} and T with their outward unit normal vectors ν_{ref} and ν and give the definition of a bijective affine transformation $\Phi : T_{\mathrm{ref}} \to T$.

(b) Given any face $F_{\mathrm{ref}} \in \mathcal{F}(T_{\mathrm{ref}})$, set $F := \Phi(F_{\mathrm{ref}}) \in \mathcal{F}(T)$. Note that $\nu_{\mathrm{ref}} \equiv \nu_{F_{\mathrm{ref}}} \in \mathbb{R}^3$ is constant on the relative interior of F_{ref} and, analogously, $\nu \equiv \nu_F$ on F. Prove that

$$|F|\,\nu_F = |F_{\mathrm{ref}}|\ \det(\mathrm{D}\Phi)(\mathrm{D}\Phi)^{-\top}\nu_{F_{\mathrm{ref}}}.$$

Hint: Use that the normal vector $\nu = e_1 \times e_2/|e_1 \times e_2|$ can be expressed by some suitable edge vectors $e_1, e_2 \in \mathbb{R}^3$ and employ the algebraic formula $(Ma) \times (Mb) = \det(M)M^{-\top}(a \times b)$ for any $M \in \mathbb{R}^{3\times 3}$ and $a, b \in \mathbb{R}^3$.

(c) Deduce that every vector field $q_{\mathrm{ref}} \in C(T_{\mathrm{ref}}; \mathbb{R}^3)$ and its Piola transform $q \in C(T; \mathbb{R}^3)$ with

$$q(x) := \frac{1}{\det(\mathrm{D}\Phi)}\mathrm{D}\Phi\, q_{\mathrm{ref}}(\Phi^{-1}(x)) \quad \text{for} \quad x \in T$$

satisfy

$$|F_{\mathrm{ref}}|\, q_{\mathrm{ref}}(\Phi^{-1}(x)) \cdot \nu_{F_{\mathrm{ref}}} = |F|\, q(x) \cdot \nu_F \quad \text{for} \quad x \in T.$$

Exercise 3 (St. Venant-Kirchhoff material). *(a)* Find a suitable counter-example to verify that $\mathbb{M}^3_+$ is no convex set.

(b) Given the two Lamé parameters $\lambda > 0$ and $\mu > 0$, consider the energy functional $\widehat{W} : \Omega \times \mathbb{M}^3_+ \to \mathbb{R}$ with

$$\widehat{W}(x, F) := \frac{\lambda}{2}(\operatorname{tr} F - 3)^2 + \mu F : F.$$

Show that this material law does *not* satisfy

$$\widehat{W}(x, F) \to +\infty \quad \text{for} \quad \det(F) \to 0.$$

(c) If $\mu < 0$, show that there exist $F \in \mathbb{M}^3_+$ such that $\widehat{W}(x, F) < 0$.

Literature. The following references concern the prerequisites from functional and numerical analysis required for this exercise sheet. Every reference is electronically available in the HU library and from HU intranet (e.g., eduroam).

- D. Braess, Finite Elemente: Theorie, schnelle Löser und Anwendungen in der Elastizitätstheorie, Springer, Berlin Heidelberg 2013.
  - Kapitel II, §1 for the definition of Sobolev spaces
  - Kapitel II, §3, Satz 3.1 for the definition of the trace operator γ_0
- B. D. Reddy, Introductory Functional Analysis With Applications to Boundary Value Problems and Finite Elements, Springer, New York 1998.
  - Chapter I, Section 5.4 for the notion of dual spaces