CPDE II, SuSe 19

Humboldt-Universität zu Berlin
Institut für Mathematik

Lecture: Prof. Fleurianne Bertrand, Tutorials: Philipp Bringmann

Exercise Sheet 3

Discussion on 14 May 2019

Please be prepared to present one or more of the following exercises on the blackboard. If not stated otherwise, algorithms or code can be displayed in Matlab or pseudocode.

Exercise 1 (Counter example to Korn's inequality for CR functions). Consider the unit square $\Omega = (-1,1)^2$ with the Dirichlet boundary $\Gamma_D := (\{-1\} \times [-1,1]) \cup ([-1,1] \times \{-1\})$ and its triangulation $\mathcal{T} = \{T_1, T_2\}$ into the two triangles

$$T_1 := \operatorname{conv}\{(-1,-1),\ (1,-1),\ (-1,1)\} \quad \text{and} \quad T_2 := \operatorname{conv}\{(-1,1),\ (1,-1),\ (1,1)\}.$$

Given $v \in H^1(\mathcal{T};\mathbb{R}^2)$, let $\mathrm{D}_{\mathrm{pw}}\, v$ denote the piecewise gradient, i.e., the function $\mathrm{D}_{\mathrm{pw}}\, v \in L^2(\Omega;\mathbb{R}^2)$ with $(\mathrm{D}_{\mathrm{pw}}\, v)|_T = \mathrm{D}(v|_T)$ for every $T \in \mathcal{T}$. Show that there exists a Crouzeix-Raviart function $v_{\mathrm{CR}} \in CR^1_0(\mathcal{T};\mathbb{R}^2)$ with

$$\varepsilon_{\mathrm{pw}}(v_{\mathrm{CR}}) = 0 \quad \text{and} \quad \mathrm{D}\, v_{\mathrm{CR}} \neq 0 \text{ in } \Omega \quad \text{and} \quad v_{\mathrm{CR}} = 0 \text{ on } \Gamma_D.$$

Can you think of a modification in order to enable the use of Crouzeix-Raviart functions for the discretisation of the linear elasticity equation?

Exercise 2 (Illustration of strain). Consider the small rectangular cuboid $\omega := (0,a_1) \times (0,a_2) \times (0,a_3)$ for $0 < a_j \ll 1$ inside the cube $\Omega := (0,1)^3$ under the deformation $\varphi : \Omega \to \mathbb{R}^3$.

(a) Show that the relative displacement of the point $a_j \mathrm{e}_j$ in x_j-direction equals the component $\varepsilon_{jj}(0)$ of the linear Green's strain tensor at the origin. *Hint:* You can neglect higher-order terms of the strain (i.e., linear deformation).

(b) If you neglect the relative displacement in x_1- and x_2-direction from part (a), the deformation φ changes the interior angle of ω in the x_1-x_2-plane at the origin (90° in reference configuration). Show that the difference approximately equals the component $2\varepsilon_{12}(0)$.

Exercise 3 (Measuring material parameters). Consider the cube $\Omega := (0,1)^3$ of some homogeneous material in a uniaxial tension test, i.e., the homogeneous stress

$$T = \begin{bmatrix} T_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

acts on Ω. By Exercise 2 (a), one can measure ε_{jj} for $j = 1, 2, 3$. The Young's modulus E and the Poisson's ratio ν are defined by

$$E := \frac{T_{11}}{\varepsilon_{11}} \quad \text{and} \quad \nu := -\frac{\varepsilon_{22}}{\varepsilon_{11}} = -\frac{\varepsilon_{33}}{\varepsilon_{11}}$$

Use the linear material law to compute the Lamé parameters λ and μ in terms of E and ν. Discuss the admissible ranges of all these parameters and look up the parameters E and ν for some materials of your choice.

Exercise 4 (Linear elasticity; implementation). Implement the solution of the linear elasticity problem in $\mathbb{R}^3$ for Lamé parameters $\lambda, \mu > 0$ with mixed boundary conditions: Seek $u \in H^1_D(\Omega; \mathbb{R}^3)$ such that

$$\int_\Omega \mathbb{C}\varepsilon(u) : \varepsilon(v)\, \mathrm{d}x = \int_\Omega f \cdot v\, \mathrm{d}x - \int_{\Gamma_N} t \cdot v\, \mathrm{d}a \quad \text{for} \quad v \in H^1_D(\Omega; \mathbb{R}^3).$$

Confirm the formulae from the Exercises 2–3. Moreover, implement a plot function for the deformed triangulation determined by $z_k = \widehat{z}_k + \gamma u(\widehat{z}_k)$. You probably need a sufficiently large scaling parameter $\gamma \gg 0$.

Hint: The software on the homepage will be updated the upcoming days to provide a function for the symmetric part of a tensor and more benchmark domains.

Literature. The following references concern the prerequisites from functional and numerical analysis required for this exercise sheet. Every reference is electronically available in the HU library and from HU intranet (e.g., eduroam).

- C. Carstensen, A young person's guide to the art of Finite Element programming, 2013.
- A. Ern and J.-L. Guermond, Theory and Practice of Finite Elements, Springer, New York, 2004.
  - Section 1.2.6 for the definition of the Crouzeix-Raviart finite element