**CPDE II, SuSe 19**
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Institut für Mathematik
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# Exercise Sheet 4

**Discussion on 21 May 2019**

Please be prepared to present one or more of the following exercises on the blackboard. If not stated otherwise, algorithms or code can be displayed in Matlab or pseudocode.

**Exercise 1 (Discrete Ladyzhenskaya lemma).** Let $\Omega \subset \mathbb{R}^2$ be a convex polygonal domain with regular triangulation $\mathcal{T}$. Given any $p_0 \in P_0(\mathcal{T}) \cap L^2_0(\Omega)$, i.e., $p_0 \in P_0(\mathcal{T})$ with $\int_\Omega p \,\mathrm{d}x = 0$, show that there exists some positive generic constant $C$ and $v_{\mathrm{CR}} \in CR^1_0(\mathcal{T};\mathbb{R}^2)$ with

$$\operatorname{div} v_{\mathrm{CR}} = p_0 \quad \text{and} \quad \| \mathrm{D}_{\mathrm{pw}}\, v_{\mathrm{CR}} \|_{L^2(\Omega)} \le C \| p_0 \|_{L^2(\Omega)}.$$

Find out, whether this result can be generalised to higher spatial dimensions and weaker regularity assumptions on $\Omega$.

*Hint:* Use the continuous version of this result from the lecture and the commutativity $\operatorname{div} \Pi_{\mathrm{CR}} = \Pi_0 \operatorname{div}$ of the canonical Crouzeix-Raviart interpolation operator.

**Exercise 2 (Divergence-free Crouzeix-Raviart basis).** Find a basis of the piecewise divergence-free Crouzeix-Raviart functions

$$Z_{\mathrm{CR}} = \{ v_{\mathrm{CR}} \in CR^1_0(\mathcal{T};\mathbb{R}^2) \ :\ \operatorname{div}_{\mathrm{pw}} v_{\mathrm{CR}} = 0 \}.$$

*Hint:* Find suitable linear independent functions and use a dimension argument. There are two kinds of divergence-free Crouzeix-Raviart functions. One kind is associated with the interior edges and are tangential to them. The second kind is associated with interior nodes and look like curls around these nodes.

**Exercise 3 (Nonlinear variational formulation).** Given some homogeneous hyperelastic material law $W : \mathbb{M}^3_+ \to \mathbb{R}$, consider the weak formulation with $u \in H^1_{\mathrm{D}}(\Omega;\mathbb{R}^3)$ such that

$$\int_\Omega \mathrm{D}\, W(\mathrm{D}\, u) : \mathrm{D}\, v \,\mathrm{d}x = \int_\Omega f \cdot v \,\mathrm{d}x - \int_{\Gamma_{\mathrm{N}}} g \cdot v \,\mathrm{d}a \quad \text{for every} \quad v \in H^1_{\mathrm{D}}(\Omega;\mathbb{R}^3).$$

Use the Galerkin method with some discrete subspace $U_h \subset H^1_{\mathrm{D}}(\Omega;\mathbb{R}^3)$ and derive a discrete formulation of this variational formulation. This system of nonlinear equations

$$A(\xi) = F$$

with the coefficient vector $\xi \in \mathbb{R}^{\dim(U_h)}$ of $u_h$ can be solved using Newton's method. Therefore, describe the computation of $A(\xi)$, $\mathrm{D}\,A(\xi)$, and $F$. Give the pseudocode of the resulting Newton's method.

**Exercise 4 (A posteriori error estimation for linear elasticity; implementation).** For the lowest-order pure displacement formulation of the linear elasticity equation from Exercise 3.4, implement the following residual-based a posteriori error estimator

$$\eta^2(\mathcal{T}, T) := \frac{1}{\mu} |T|^{2/3} \|f\|^2_{L^2(T)} + \mu\, |T|^{1/3} \sum_{F \in \mathcal{F}(T) \setminus \mathcal{F}(\Gamma_{\mathrm{D}})} \|[\sigma_h \nu]_F\|^2_{L^2(E)}$$

with $\sigma_h := \mathbb{C}\varepsilon(u_{\mathrm{C}})$ in the jump

$$[\sigma_h \nu]_F = \begin{cases} (\sigma_h|_{T_+} - \sigma_h|_{T_-})|_F\, \nu & \text{if } F \in \mathcal{F}(\Omega), \\ \sigma_h\, \nu - t & \text{if } F \in \mathcal{F}(\Gamma_{\mathrm{N}}). \end{cases}$$

Add the error estimation to your existing code and create a convergence history plot. *Hint:* The code for the convergence history plot will be provided on the lecture's homepage the upcoming days.

**Literature.** The following references concern the prerequisites from functional and numerical analysis required for this exercise sheet. Every reference is electronically available in the HU library and from HU intranet (e.g., eduroam).

- C. Carstensen, A young person's guide to the art of Finite Element programming, 2013.
- A. Ern and J.-L. Guermond, Theory and Practice of Finite Elements, Springer, New York, 2004.
  - Section 1.2.6 for the definition of the Crouzeix-Raviart finite element
- R. Verfürth, A Review of A Posteriori Error Estimation Techniques for Elasticity Problems, In: P. Ladevèze and J. T. Oden (Eds.), Advances in Adaptive Computational Methods in Mechanics, vol. 47, Elsevier Science, 1998.
  - for the definition of the residual based a posteriori error estimator for linear elasticity