CPDE II, SuSe 19
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Exercise Sheet 5

Discussion on 28 May 2019

Please be prepared to present one or more of the following exercises on the blackboard. If not stated otherwise, algorithms or code can be displayed in Matlab or pseudocode.

Exercise 1 (Variational formulation of least-squares FEM). *(a)* Derive the necessary optimality condition of first-order for the minimisation of the general least-squares functional

$$LS(f;u) := \|f - Qu\|_U^2.$$

What happens if Q is a second-order differential operator?

(b) Apply this to the least-squares functional of the linear elasticity

$$LS(f;\sigma,u) := \|f + \operatorname{div}\sigma\|_{L^2(\Omega)}^2 + \|\mathbb{C}^{-1}\sigma - \varepsilon(u)\|_{L^2(\Omega)}^2.$$

(c) Use the homogeneous hyperelastic material law $W : \mathbb{M}^3_+ \to \mathbb{R}$ from Exercise 4.3 in the least-squares functional

$$LS(f;\sigma,u) := \|f + \operatorname{div}\sigma\|_{L^2(\Omega)}^2 + \|\sigma - \mathrm{D}\,W(\varepsilon(u))\|_{L^2(\Omega)}^2$$

and formally derive the associated variational formulation. What is the crucial difference between this formulation and the one from Exercise 4.3?

Exercise 2 (Symmetric Raviart-Thomas functions). Abbreviate $\mathbb{S} := \mathbb{R}^{3\times 3}_{\mathrm{sym}}$. Prove that

$$\Sigma_h := \{\tau_h \in RT_0(\mathcal{T}_h;\mathbb{R}^{3\times 3}) \ : \ \operatorname{as}\tau_h = 0\} \subset H(\operatorname{div} = 0, \Omega; \mathbb{S})$$

Construct a counter-example to show that there exists no positive generic constant C with

$$\inf_{\tau_h\in\Sigma_h} \|\sigma - \tau_h\|_{H(\operatorname{div},\Omega)} \le Ch\Big(|\sigma|_{H^1(\Omega)} + |\operatorname{div}\sigma|_{H^1(\Omega)}\Big) \quad \text{for every} \quad \sigma \in H^1(\Omega;\mathbb{S}).$$

Exercise 3 (Vector potentials). Let $\Omega \subset \mathbb{R}^3$ be a simply-connected Lipschitz domain. Prove that $\rho \in H(\operatorname{div} = 0, \Omega)$ if and only if there exists a vector potential $\beta \in H^1(\Omega;\mathbb{R}^3)$

$$\operatorname{curl}\beta = \rho \quad \text{and} \quad \operatorname{div}\beta = 0. \tag{$*$}$$

Hint: Extend ρ by a Neumann problem to a ball containing Ω and then by zero to the whole space $\mathbb{R}^3$. Characterise $\operatorname{div}\rho = 0$ and ($*$) in terms of the Fourier transform $\mathscr{F}$ as

$$\mathscr{F}(\operatorname{div}\rho) = 0, \quad \mathscr{F}(\operatorname{curl}\beta) = \mathscr{F}\rho \quad \text{and} \quad \mathscr{F}(\operatorname{div}\beta) = 0.$$

To this end, use the formula $\mathscr{F}(\mathrm{D}^\alpha v)(\xi) = \mathrm{i}^{|\alpha|}\xi^\alpha \mathscr{F}v(\xi)$ and solve the algebraic equations.

Exercise 4 (Helmholtz decomposition). Let $\Omega \subset \mathbb{R}^3$ be a simply-connected Lipschitz domain. Given $f \in L^2(\Omega; \mathbb{R}^3)$, show that there exist $\alpha \in H^1(\Omega)/\mathbb{R}$ and $\beta \in H^1(\Omega)$ such that

$$f = \nabla\alpha + \operatorname{curl}\beta.$$

Are α and β unique? Is this decomposition orthogonal?

Hint: Determine α by some auxiliary problem such that $f - \nabla\alpha$ is divergence-free (in a weak sense) and apply Exercise 2.

Exercise 5 (A posteriori error estimation for linear elasticity; implementation). We will discuss the solution of Exercise 4.4.

Literature. The following references concern the prerequisites from functional and numerical analysis required for this exercise sheet. Every reference is electronically available in the HU library and from HU intranet (e.g., eduroam).

- C. Carstensen, A young person's guide to the art of Finite Element programming, 2013.
- D. Werner, Funktionalanalysis, Springer, Berlin, 2018.
  - Abschnitt V.2 for definition and results on Fourier transform for Sobolev functions