CPDE II, SuSe 19
Humboldt-Universität zu Berlin
Institut für Mathematik
Lecture: Prof. Fleurianne Bertrand, Tutorials: Philipp Bringmann

Exercise Sheet 6

Discussion on 3 June 2019

Please be prepared to present one or more of the following exercises on the blackboard. If not stated otherwise, algorithms or code can be displayed in Matlab or pseudocode.

Exercise 1 (Ellipticity of LS functional). Prove the ellipticity of the least-squares functional $\mathcal{F}_{\mathbb{C}^{-1}} : H_{\mathrm{N}}(\operatorname{div}, \Omega, \mathbb{R}^{3\times 3}) \times H^1_{\mathrm{D}}(\Omega; \mathbb{R}^3) \times L^2(\Omega) \to \mathbb{R}$ given by

$$\mathcal{F}_{\mathbb{C}^{-1}}(\sigma, u; f) = \|f + \operatorname{div} \sigma\|^2_{L^2(\Omega)} + \|\mathbb{C}^{-1}\sigma - \varepsilon(u)\|^2_{L^2(\Omega)}$$

with respect to the norm

$$|||(\sigma, u)|||^2 = \|\sigma\|^2_{H(\operatorname{div},\Omega)} + \|u\|^2_{H^1(\Omega)}.$$

Justify that the generic constant is independent from the Lamé parameter λ.

Hint: Use the tr-dev-div inequality from the lecture.

Exercise 2 (Best-approximation results for LS FEM). *(a)* Derive Céa's lemma for the least-squares FEM from the previous exercise.

(b) Given that the exact solution is sufficiently smooth, provide the order of convergence (on quasi-uniform meshes) for various choices of conforming discretisations, e.g., RT_k, BDM_k, $BDFM_k$, AW_k, DP_k, P_k. Assess the quality of the discretisations.

Hint: Find the corresponding interpolation error estimates in the literature.

Exercise 3 (Arnold-Winther FEM). Let $T = \operatorname{conv}\{z_1, z_2, z_3\} \in \mathcal{T}$ denote some non-degenerated triangle with edges $\mathcal{E}(T) = \{E_1, E_2, E_3\}$. The Arnold-Winther mixed finite element on T consists of the components $(T, \Sigma_T, \mathcal{K})$ and $(T, U_T, \mathcal{L})$ given by the finite dimensional spaces

$$\Sigma_T := P_2(T; \mathbb{S}) + \{\sigma_h \in P_3(T; \mathbb{S}) \ : \ \operatorname{div} \sigma_h = 0\} \quad \text{and} \quad U_T = P_1(T; \mathbb{R}^2)$$

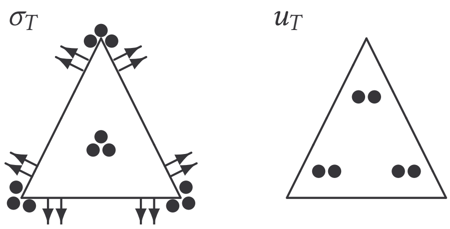
and the linear functionals

$$\begin{aligned}\mathcal{K} := \{&\sigma \mapsto \sigma_{11}(z_1), \quad \sigma \mapsto \sigma_{12}(z_1), \quad \sigma \mapsto \sigma_{22}(z_1),\\ &\sigma \mapsto \sigma_{11}(z_2), \quad \sigma \mapsto \sigma_{12}(z_2), \quad \sigma \mapsto \sigma_{22}(z_2),\\ &\sigma \mapsto \sigma_{11}(z_3), \quad \sigma \mapsto \sigma_{12}(z_3), \quad \sigma \mapsto \sigma_{22}(z_3),\\ &\sigma \mapsto \fint_{E_1} (\sigma\nu_{E_1})_1 \,\mathrm{d}a, \quad \sigma \mapsto \fint_{E_1} (\sigma\nu_{E_1})_2 \,\mathrm{d}a, \quad \sigma \mapsto \fint_{E_1} \theta_1(\sigma\nu_{E_1})_1 \,\mathrm{d}a, \quad \sigma \mapsto \fint_{E_1} \theta_1(\sigma\nu_{E_1})_2 \,\mathrm{d}a,\\ &\sigma \mapsto \fint_{E_2} (\sigma\nu_{E_2})_1 \,\mathrm{d}a, \quad \sigma \mapsto \fint_{E_2} (\sigma\nu_{E_2})_2 \,\mathrm{d}a, \quad \sigma \mapsto \fint_{E_2} \theta_2(\sigma\nu_{E_2})_1 \,\mathrm{d}a, \quad \sigma \mapsto \fint_{E_2} \theta_2(\sigma\nu_{E_2})_2 \,\mathrm{d}a,\\ &\sigma \mapsto \fint_{E_3} (\sigma\nu_{E_3})_1 \,\mathrm{d}a, \quad \sigma \mapsto \fint_{E_3} (\sigma\nu_{E_3})_2 \,\mathrm{d}a, \quad \sigma \mapsto \fint_{E_3} \theta_3(\sigma\nu_{E_3})_1 \,\mathrm{d}a, \quad \sigma \mapsto \fint_{E_3} \theta_3(\sigma\nu_{E_3})_2 \,\mathrm{d}a,\\ &\sigma \mapsto \fint_T \sigma_{11} \,\mathrm{d}x, \quad \sigma \mapsto \fint_T \sigma_{12} \,\mathrm{d}x, \quad \sigma \mapsto \fint_T \sigma_{22} \,\mathrm{d}x\}\end{aligned}$$

with the tangential function $\theta_j(x) := (x - \operatorname{mid}(E_j))\tau_{E_j}$ as well as

$$\mathcal{L} := \{u \mapsto u_1(z_1), \quad u \mapsto u_2(z_1), \quad u \mapsto u_1(z_2), \quad u \mapsto u_2(z_2), \quad u \mapsto u_1(z_3), \quad u \mapsto u_2(z_3)\}.$$

In short,



(a) Compute $\dim \Sigma_T$ and $\dim U_T$.

(b) Prove unisolvence (i.e., linear independence) of the functionals in $\mathcal{K}$ and $\mathcal{L}$.

(c) Present a strategy to compute the dual basis functions on the reference triangle.

Literature. The following references concern the prerequisites from functional and numerical analysis required for this exercise sheet. Every reference is electronically available in the HU library and from HU intranet (e.g., eduroam).

- F. Brezzi, M. Fortin, Mixed and Hybrid Finite Element Methods, Springer, New York, 1991.
  - Section III.3 for the definition and interpolation error estimates of various $H(\mathrm{div}, \Omega)$-conforming discretisations

  (If available, use the revisited version of this book instead: *Mixed Finite Element Methods and Applications* by Boffi, Brezzi, Fortin from 2013)
- D. Arnold, R. Winther, Mixed finite elements for elasticity, Numer. Math. 92, 2002, p. 401–419.
  - Section 3 for the definition of the Arnold-Winther element