CPDE II, SuSe 19
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Exercise Sheet 9

Discussion on 25 June 2019

Please be prepared to present one or more of the following exercises on the blackboard. If not stated otherwise, algorithms or code can be displayed in Matlab or pseudocode.

Exercise 1 (Vector potentials). We will discuss the solution to Exercise 3 on Sheet 5.

Exercise 2 (Fortin criterion). Let V, Q be Hilbert spaces, $V_h \subset V$ and $Q_h \subset Q$ subspaces of V and Q, respectively, and $b : V \times Q \to \mathbb{R}$ a continuous bilinear form satisfying the inf-sup condition

$$0 < \beta \leq \inf_{q \in Q \setminus \{0\}} \sup_{v \in V \setminus \{0\}} \frac{b(v, q)}{\|v\|_V \|q\|_Q}.$$

Prove that the following two statements are equivalent.

(a) V_h and Q_h satisfy the discrete inf-sup condition, i. e.

$$0 < \beta_h \leq \inf_{q_h \in Q_h \setminus \{0\}} \sup_{v_h \in V_h \setminus \{0\}} \frac{b(v_h, q_h)}{\|v_h\|_V \|q_h\|_Q}.$$

(b) There exists a linear bounded operator $\Pi_h : V \to V_h$ such that any $v \in V$ satisfies $b(v - \Pi_h v, q_h) = 0$ for any $q_h \in Q_h$.

Hint: The more tricky part is the direction $(a) \implies (b)$. To define Π_h consider the auxiliary saddle point problem

$$\begin{aligned} (v_h, w_h)_V + b(w_h, q_h) &= (v, w_h)_V \quad \text{for any } w_h \in V_h, \\ b(v_h, r_h) &= b(v, r_h) \quad \text{for any } r_h \in Q_h \end{aligned}$$

and use Brezzi's splitting theorem (Theorem 6.1 from the lecture) to obtain existence and uniqueness of a solution.

Exercise 3 (Clément interpolation). The Clément interpolation operator $I_h : H^1_0(\Omega) \to S^1_0(\mathcal{T})$ ($S^1_0(\mathcal{T}) = \mathcal{P}_1(\mathcal{T}) \cap H^1_0(\Omega)$) is defined for any $v \in H^1_0(\Omega)$ via the node values

$$I_h v(z) = \begin{cases} 0 & \text{if } z \in \mathcal{N}(\partial\Omega), \\ \frac{1}{|\omega_z|} \int_{\omega_z} v \, \mathrm{d}x & \text{if } z \in \mathcal{N}(\Omega) \end{cases}$$

and satisfies for any $T \in \mathcal{T}$

(a) $\|v - I_h v\|_{L^2(T)} \lesssim h_T \|\nabla v\|_{L^2(\omega_T)}$,

(b) $\|\nabla(v - I_h v)\|_{L^2(T)} \lesssim \|\nabla v\|_{L^2(\omega_T)}$.

The arguments from the lecture hold for any $T \in \mathcal{T}$ with no boundary nodes, i.e. $\mathcal{N}(T) \cap \mathcal{N}(\partial\Omega) = \emptyset$. Prove (a) and (b) for the case $\mathcal{N}(T) \cap \mathcal{N}(\partial\Omega) \neq \emptyset$.

Literature. The following references concern the prerequisites from functional and numerical analysis required for this exercise sheet. Every reference is electronically available in the HU library and from HU intranet (e.g., eduroam).

- F. Brezzi, M. Fortin, Mixed and Hybrid Finite Element Methods, Springer, New York, 1991.
  - Section IIII.2 for Brezzi's splitting theorem

  (If available, use the revisited version of this book instead: *Mixed Finite Element Methods and Applications* by Boffi, Brezzi, Fortin from 2013)