CPDE II, SuSe 19
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Exercise Sheet 10

Discussion on 02 July 2019

Please be prepared to present one or more of the following exercises on the blackboard.

Exercise 1 (Spurious pressure modes). In the setting of Exercise 2 on Sheet 9, define $B^T : Q \to V^*$ and $B_h^T : Q_h \to V_h^*$ with

$$B^T q(v) = b(v, q) \quad \text{for any } v \in V,$$
$$B_h^T q_h(v_h) = b(v_h, q_h) \quad \text{for any } v_h \in V_h.$$

Set $S_h = \operatorname{Ker} B_h^T \setminus \operatorname{Ker} B^T$. Prove that the discrete inf-sup condition implies $S_h = \emptyset$.

Exercise 2 (CR-P0 elements for pure homogenous Dirichlet boundary). Let $\Omega \subset \mathbb{R}^2$ be bdd. polyhedral Lipschitz domain, $V_h = (\mathrm{CR}_0^1(\mathcal{T}))^2$, endowed with the norm $\|v_h\|_{V_h} = \|\mathrm{D}_{\mathrm{pw}} v_h\|_{L^2(\Omega)}$, and $Q_h = P_0(\mathcal{T}) \cap L_0^2(\Omega)$.

(a) Prove the discrete inf-sup condition

$$0 < \beta_h \le \inf_{q_h \in Q_h} \sup_{v_h \in V_h} \frac{(\operatorname{div} v_h, q_h)_{L^2(\Omega)}}{\|v_h\|_{V_h} \|q_h\|_Q}.$$

Hint: Use the standard nonconforming operator $I_{\mathrm{NC}} : H_0^1(\Omega) \to \mathrm{CR}_0^1(\mathcal{T})$ for the Fortin interpolation.

(b) Prove that the discrete mixed formulation, find $u_h \in V_h$ and $p_h \in Q_h$ such that

$$(\mathrm{D}_{\mathrm{pw}} u_h, \mathrm{D}_{\mathrm{pw}} v_h)_{L^2(\Omega)} - (\operatorname{div}_{\mathrm{pw}} v_h, p_h)_{L^2(\Omega)} = (\Pi_0 f, v_h)_{L^2(\Omega)} \quad \text{for any } v_h \in V_h,$$
$$(\operatorname{div}_{\mathrm{pw}} u_h, q_h)_{L^2(\Omega)} = 0 \quad \text{for any } q_h \in Q_h,$$

has a unique solution and derive $\operatorname{div}_{\mathrm{pw}} u_h = 0$ a.e. in Ω.

(c) Suppose that $u \in (H^2(\Omega))^2 \cap H_0^1(\Omega)$ and $p \in H^1(\Omega) \cap L_0^2(\Omega)$. Prove the optimal convergence order

$$\|\mathrm{D}_{\mathrm{pw}}(u - u_h)\|_{L^2(\Omega)} + \|p - p_h\|_{L^2(\Omega)} \lesssim h\|u\|_{H^2(\Omega)} + \operatorname{osc}(f, \mathcal{T}),$$

where (u, p) is the weak solution to

$$\begin{aligned} -\Delta u + \nabla p &= f \quad \text{in } \Omega, \\ \operatorname{div} u &= 0 \quad \text{in } \Omega. \end{aligned}$$

Hint: You can use the estimate from [Boffi-Brezzi-Fortin Proposition 5.5.6]

$$\begin{aligned} &\| \mathrm{D}_{\mathrm{pw}}(u - u_h)\|_{L^2(\Omega)} + \|p - p_h\|_{L^2(\Omega)} \\ &\quad \lesssim \inf_{v_h \in V_h} \| \mathrm{D}_{\mathrm{pw}}(u - v_h)\|_{L^2(\Omega)} + \inf_{q_h \in Q_h} \|p - q_h\|_{L^2(\Omega)} \\ &\qquad + \|(f, \bullet)_{L^2(\Omega)} - (\Pi_0 f, \bullet)_{L^2(\Omega)}\|_{V_h^*} + \|(\mathrm{D}\, u, \mathrm{D}_{\mathrm{pw}} \bullet) - (\operatorname{div}_{\mathrm{pw}} \bullet, p) - (f, \bullet)\|_{V_h^*}. \end{aligned}$$

Note that the dual norm of V_h is defined by

$$\|g\|_{V_h^*} = \sup_{v_h \in V_h} g(v_h)/\|v_h\|_{V_h}.$$

Literature. The following references concern the prerequisites from functional and numerical analysis required for this exercise sheet. Every reference is electronically available in the HU library and from HU intranet (e.g., eduroam).

- D. Boffi, F. Brezzi, M. Fortin, Mixed Finite Element Methods and Applications, Springer Series in Computational Mathematics 44, 2013