CPDE II, SuSe 19
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Exercise Sheet 11

Discussion on 09 July 2019

Please be prepared to present one or more of the following exercises on the blackboard.

Exercise 1 (Crouzeix-Raviart typed functions of second order). Recall from the lecture the spaces $P^2_{\mathrm{NC}}(\mathcal{T})$, $B^2_{\mathrm{NC}}(\mathcal{T})$ and $\mathcal{P}^{2\setminus 1}(\mathcal{T})$.

(a) Compute $\dim(B^2_{\mathrm{NC}}(\mathcal{T}))$.

(b) Prove the split $P^2_{\mathrm{NC}}(\mathcal{T}) = \mathcal{P}^2(\mathcal{T}) \oplus B^2_{\mathrm{NC}}(\mathcal{T}) = \mathcal{P}^1(\mathcal{T}) \oplus \mathcal{P}^{2\setminus 1}(\mathcal{T}) \oplus B^2_{\mathrm{NC}}(\mathcal{T})$.

Exercise 2 (Strang's first lemma). Consider the variational formulation, find $u \in V$

$$a(u, v) = f(v) \quad \text{for any } v \in V$$

with a Hilbert space V, a V-elliptic bounded bilinear form a and RHS $f \in V^*$. For $V_h \subset V$, suppose that $a_h : V_h \times V_h \to \mathbb{R}$ (resp. $f_h : V_h \to \mathbb{R}$) is a V_h-elliptic bounded bilinear form (resp. bounded linear functional). Prove that the solution $u_h \in V_h$ to the discrete problem

$$a_h(u_h, v_h) = f(v_h) \quad \text{for any } v_h \in V_h$$

satisfies the a priori estimate

$$\|u - u_h\|_V \lesssim \inf_{v_h \in V_h} \left(\|u - v_h\|_V + \sup_{w_h \in V_h} \frac{a(v_h, w_h) - a_h(v_h, w_h)}{\|w_h\|_V} \right) + \sup_{w_h \in V_h} \frac{f(w_h) - f_h(w_h)}{\|w_h\|_V}.$$

Exercise 3 (Strang's second lemma for PMP). Let $u \in H^1_0(\Omega)$ solve the weak formulation of the Poisson model problem $-\Delta u = f$ with some RHS $f \in L^2(\Omega)$. For the nonconforming discretization $\mathrm{CR}^1_0(\mathcal{T}) \not\subseteq H^1_0(\Omega)$, suppose that $u_h \in \mathrm{CR}^1_0(\mathcal{T})$ solve the discrete problem

$$\int_\Omega \nabla_{\mathrm{pw}} u_h \cdot \nabla_{\mathrm{pw}} v_h \,\mathrm{d}x = \int_\Omega f v_h \,\mathrm{d}x \quad \text{for any } v_h \in \mathrm{CR}^1_0(\mathcal{T}).$$

Prove that

$$\|\nabla_{\mathrm{pw}}(u - u_h)\|_{L^2(\Omega)} \lesssim \inf_{v_h \in \mathrm{CR}^1_0(\mathcal{T})} \|\nabla_{\mathrm{pw}}(u - v_h)\|_{L^2(\Omega)} + \sup_{w_h \in \mathrm{CR}^1_0(\mathcal{T})} \frac{(\nabla u, \nabla_{\mathrm{pw}} w_h) - (f, w_h)}{\|\nabla_{\mathrm{pw}} w_h\|_{L^2(\Omega)}}.$$