Symmetry breaking operators for reductive pairs

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Workshop on Conformal geometry and Spectral Theory
on the occasion of Andreas Juhl's 60^{th} birthday
Berlin, November 12, 2016

Standard questions in Representation Theory

A representation of a group G is a specific way to realize G by linear transformations (symmetries) on some vector spaces.

1. Classification of elementary blocks (irreducible representations) and the unitary dual $\widehat{G}$.
   a. $\mathfrak{S}_n$ - Partitions. Compact Lie group - Borel-Weil-Bott theorem.
   b. Nilpotent Lie group - Kirillov's Orbit method.
   c. Real reductive Lie group - parabolic, cohomological induction, Harish-Chandra theory. No general understanding of $\widehat{G}$.
2. Branching rules $G \downarrow G'$:

$$\pi\big|_{G'} = \int_{\widehat{G'}} m(\pi,\nu)\nu d\mu(\nu), \quad m(\pi,\nu) : \widehat{G'} \to \mathbb{N} \cup \{\infty\}.$$

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Example 1. The Abelian compact Lie group $G_1 = G_1' = SO(2) \simeq S^1 \simeq \mathbb{R}/2\pi\mathbb{Z}$ acts (reducibly) on $L^2(S^1)$ by $(L(g)f)(h) = f(g^{-1}h)$, with $g, h \in SO(2)$.

Plancherel Theorem for square integrable 2π-periodic functions says

$$L = \sum_{n\in\mathbb{Z}}^{\oplus} \chi_n,$$

where $\chi_n : G_1 \to \mathbb{C}^\times$ is given by $\chi_n(e^{i\phi}) = e^{in\phi}$. Uniqueness of the Fourier coefficients $\Leftrightarrow$ $m(L,\chi_n) = 1$ for every $n \in \mathbb{Z}$ ($\simeq \widehat{G_1}$).

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Example 2. Similarly, for the Abelian noncompact Lie group $G_2 = G_2' = \mathbb{R}$ we have

$$L \simeq \int_{\widehat{\mathbb{R}}\simeq\mathbb{R}}^{\oplus} \chi_\lambda d\lambda,$$

where $\chi_\lambda : G_2 \to \mathbb{C}^\times$ is given by $\chi_\lambda(x) = e^{2i\pi x\lambda}$, with $\lambda \in \mathbb{R} \simeq \widehat{G}_2$.

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$$(\pi_k(g)f))(z) = (cz+d)^{-k} f\left(\frac{az+b}{cz+d}\right),$$

where $g^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{R})$ and $\Pi = \{z = x+iy, x \in \mathbb{R}, y > 0\} \simeq SL(2,\mathbb{R})/SO(2)$

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Abstract branching (fusion) rule. [V. Molchanov 1979]

The branching rule for the tensor product of two holomorphic discrete series representations of $SL(2,\mathbb{R})$ is given by :

$$\pi_{k_1} \otimes \pi_{k_2} \simeq \sum_{a\in\mathbb{N}}^{\oplus} \pi_{k_1+k_2+2a}. \quad (1)$$

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Rankin-Cohen brackets on the upper half-plane Π

$$RC^a_{k_1,k_2}(f_1,f_2)(z) = \sum_{j=0}^{a}(-1)^j \binom{k_1+a-1}{j}\binom{k_2+a-1}{a-j} f_1^{(a-j)}(z) f_2^{(j)}(z),$$

where $f_j \in \mathcal{H}^2_{k_j}(\Pi), j = 1,2$ and $f^{(\ell)}(z) := \frac{\partial^\ell f}{\partial z^k}$.

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Equivariance

The $\mathrm{RC}^a_{k_1,k_2}$ are intertwining operators for the abstract branching rule (1), i.e.

$$\mathrm{RC}^a_{k_1,k_2}(\pi_{k_1}(g)f_1, \pi_{k_2}(g)f_2) = \pi_{k_1+k_2+2a}(g)\mathrm{RC}^a_{k_1,k_2}(f_1,f_2).$$

for every $a \in \mathbb{N}, g \in SL(2,\mathbb{R}), f_j \in \mathcal{H}^2_{k_j}(\Pi), j = 1,2$.

Rankin-Cohen brackets-motivations

Numerous applications :

- Explicit construction of holomorphic modular forms.
- Modular and quasimodular forms (special values of L-functions, the Ramanujan and Chazy differential equations, van der Pol and Niebur equalities).
- Covariant quantization, noncommutative geometry, cyclic cohomology.
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Why $$(-1)^j \binom{k_1 + a - 1}{j} \binom{k_2 + a - 1}{a - j}?$$

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- There exists several explicit constructions of $\mathrm{RC}^{a}_{k_1,k_2}$:
  - Recursions (Howe).
  - Taylor coefficients for Jacobi forms (Eichler-Zagier).
  - Reproducing kernels for Hilbert spaces (Zhang).
  - Dual pairs correspondence (Ibukiyama).
- Transvectants (Überschiebungen), Cayley Ω-process.
- Notice that Rankin-Cohen operators are differential operators.
- New and broader approach : F-method based on branching rules and *symmetry breaking operators* for symmetric pairs, i.e. $(G' = G^{\sigma} = \{g \in G : \sigma(g) = g\}$ for a certain involution σ of G).

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If $Y \subset X$ the restrictions of normal derivatives are G'-equivariant diff. operators.
How to describe $\mathrm{Diff}_{G'}(\mathcal{V}_X, \mathcal{W}_Y) \subset \mathrm{Hom}_{G'}(\mathcal{O}(X, \mathcal{V}), \mathcal{O}(Y, \mathcal{W}))$?

Black box

F-method

- Algebraic Fourier Transform for generalized Verma modules.
- Orbits reduction.
- T-saturation for equvariant sheaves of $\mathcal{D}$-modules.

Algebraic Fourier Transform for generalized Verma modules

F-method

Let $G \supset G'$ be a pair of real reductive Lie groups, and $P \supset P'$ a pair of parabolic subgroups with compatible Levi decompositions $P = LN_+ \supset P' = L'N'_+$ such that $L \supset L'$ and $N_+ \supset N'_+$. Let (σ_λ, V) and (τ_ν, W) be finite-dimensional representations of P and P' with trivial actions of N_+ and N'_+, respectively.

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1. *(duality)* There is a natural isomorphism :

$$D_{X\to Y}\colon \mathrm{Hom}_{\mathfrak{g}',P'}(\mathrm{ind}_{\mathfrak{p}'}^{\mathfrak{g}'}(W^\vee), \mathrm{ind}_{\mathfrak{p}}^{\mathfrak{g}}(V^\vee)) \xrightarrow{\sim} \mathrm{Diff}_{G'}(\mathcal{V}_X, \mathcal{W}_Y).$$

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2. *(extension)* The restriction $\mathcal{W}_Z|_Y \simeq \mathcal{W}_Y$ with $(Z = G/P')$ induces the bijection

$$\mathrm{Rest}_Y \colon \mathrm{Diff}_G(\mathcal{V}_X, \mathcal{W}_Z) \xrightarrow{\sim} \mathrm{Diff}_{G'}(\mathcal{V}_X, \mathcal{W}_Y).$$

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For $\psi \in (\mathrm{Pol}(\mathfrak{n}_+) \otimes V^\vee) \otimes W \simeq \mathrm{Hom}_{\mathbb{C}}(V, W \otimes \mathrm{Pol}(\mathfrak{n}_+))$, consider a system of partial differential equations

$$(\widehat{d\pi_{(\sigma,\lambda)^*}}(C) \otimes \mathrm{id}_W)\psi = 0 \quad \text{for all } C \in \mathfrak{n}'_+, \tag{2}$$

and set

$$Sol(\mathfrak{n}_+; \sigma_\lambda, \tau_\nu) := \{\psi \in \mathrm{Hom}_{L'}(V, W \otimes \mathrm{Pol}(\mathfrak{n}_+)) : \psi \text{ solves } (2)\}.$$

Then there is a natural isomorphism

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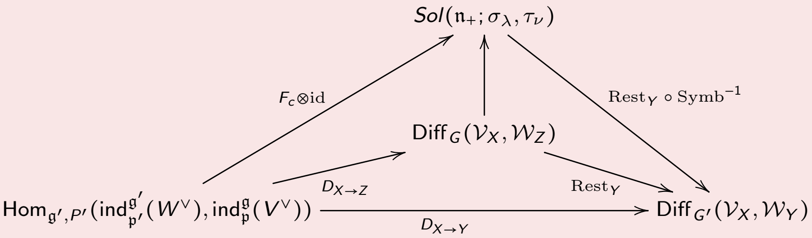
Algebraic Fourier Transform for generalized Verma modules

F-method (T. Kobayashi-P. 2015)

Assume that the nilradical $\mathfrak{n}_+$ is abelian. Then, the system (2) is of second order, and the following diagram commutes :



Revealing examples : hermitian symmetric spaces

- T. Kobayashi-P. 2015. Part II.
  1. G/K is Hermitian sym. space.
  2. $\dim V = 1$.
  3. V and W are irreducible.
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  That is, we consider equivariant embeddings $Y \hookrightarrow X$ in 6 different geometries :

1.	$\mathbb{P}^n\mathbb{C} \hookrightarrow \mathbb{P}^n\mathbb{C} \times \mathbb{P}^n\mathbb{C}$	4.	$\mathrm{Gr}_{p-1}(\mathbb{C}^{p+q}) \hookrightarrow \mathrm{Gr}_p(\mathbb{C}^{p+q})$
2.	$\mathrm{LGr}(\mathbb{C}^{2n-2}) \times \mathrm{LGr}(\mathbb{C}^2) \hookrightarrow \mathrm{LGr}(\mathbb{C}^{2n})$	5.	$\mathbb{P}^n\mathbb{C} \hookrightarrow \mathrm{Q}^{2n}\mathbb{C}$
3.	$\mathrm{Q}^n\mathbb{C} \hookrightarrow \mathrm{Q}^{n+1}\mathbb{C}$	6.	$\mathrm{IGr}_{n-1}(\mathbb{C}^{2n-2}) \hookrightarrow \mathrm{IGr}_n(\mathbb{C}^{2n})$

Revealing examples : hermitian symmetric spaces

The equivariant embeddings $Y \hookrightarrow X$:

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correspond to six symmetric pairs (G, G') :

1. $(U(n,1) \times U(n,1), U(n,1))$
2. $(Sp(n,\mathbb{R}), Sp(n-1,\mathbb{R}) \times Sp(1,\mathbb{R}))$
3. $(SO(n,2), SO(n-1,2))$
4. $(SU(p,q), S(U(1) \times U(p-1,q)))$
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Theorem (T. Kobayashi-P. 2015)

(1) Any G'-intertwining operator from $\mathcal{O}(X, \mathcal{L}_\lambda)$ to $\mathcal{O}(Y, \mathcal{W})$ is given by normal derivatives with respect to the equivariant embedding $Y \hookrightarrow X$ of type (4), (5) or (6).
(2) None of normal derivatives of positive order is a G'-intertwining operator for $Y \hookrightarrow X$ of type (1), (2) and (3).

Examples

In this situation the system of PDE on symbols of symmetry breaking operators

$$\widehat{d\pi_\mu}(\mathfrak{n}'_+)\psi = 0$$

reduces, by the method of *T-saturation* of the underlying $\mathcal{D}$-modules to the Gauss hypergeometric equation :

$$\left(z(1-z)\frac{d^2}{dz^2} - (c-(a+b+1)z)\frac{d}{dz} - ab \right) u(z) = 0.$$

Answers

- Let

$$C_\ell^\alpha(t) = \sum_{k=0}^{\left[\frac{\ell}{2}\right]}(-1)^k\frac{\Gamma(\ell-k+\alpha)}{\Gamma(\alpha)\Gamma(k+1)\Gamma(\ell-2k+1)}(2t)^{\ell-2k}.$$

  be the Gegenbauer polynomial.

- and

$$P_\ell^{\alpha,\beta}(t) = \frac{\Gamma(\alpha+\ell+1)}{\Gamma(\alpha+\beta+\ell+1)}\sum_{m=0}^{\ell}\binom{\ell}{m}\frac{\Gamma(\alpha+\beta+\ell+m+1)}{\ell!\Gamma(\alpha+m+1)}\left(\frac{t-1}{2}\right)^m.$$

  the Jacobi polynomial.

- Let

$$(I_\ell\widetilde{C}_\ell^\mu) := x^{\frac{\ell}{2}}C_\ell^\alpha\left(\frac{y}{\sqrt{x}}\right) = \sum_{k=0}^{\left[\frac{\ell}{2}\right]}(-1)^k\frac{\Gamma(\ell-k+\alpha)}{\Gamma(\alpha)\Gamma(k+1)\Gamma(\ell-2k+1)}(2y)^{\ell-2k}x^k.$$

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Answers

Let $\mathcal{L}_\lambda$ be a homogeneous line bundle and $\mathcal{W}^a_\lambda$ a homogeneous vector bundle $G^\tau \times_{P'} (S^a(\mathfrak{n}^\tau_+) \otimes \mathbb{C}_\lambda)$.

Theorem D1 : $(U(n,1) \times U(n,1), U(n,1))$

The differential operator

$$D_{X \to Y,a} := (I_a P_a^{\lambda'-1, -\lambda'-\lambda''-2a+1}) \left(\sum_{i=1}^{n} v_i \frac{\partial}{\partial z_i}, \sum_{j=1}^{n} v_j \frac{\partial}{\partial z_j} \right)$$

intertwines $\mathcal{O}(Y, \mathcal{L}_{(\lambda'_1,\lambda'_2)}) \otimes \mathcal{O}(Y, \mathcal{L}_{(\lambda''_1,\lambda''_2)})$ with $\mathcal{O}(Y, \mathcal{W}^a_{(\lambda'_1+\lambda''_1, \lambda'_2+\lambda''_2)})$, where $\lambda'_1, \lambda''_1, \lambda'_2, \lambda''_2 \in \mathbb{Z}$, $\lambda' = \lambda'_1 - \lambda'_2$, $\lambda'' = \lambda''_1 - \lambda''_2$, and $a \in \mathbb{N}$.

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If $n = 1$ one recovers the Rankin-Cohen brackets :

$$\mathcal{RC}^a_{\lambda',\lambda''} = (-1)^a P_a^{\lambda'-1,1-\lambda'-\lambda''-2a}\left(\frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2}\right)\Big|_{z_1=z_2=z}.$$

Answers

Theorem D2 : $(Sp(n,\mathbb{R}), Sp(n-1,\mathbb{R}) \times Sp(1,\mathbb{R}))$

The differential operator

$$D_{X\to Y,a} := (I_a C_a^{\lambda-1})\left(\sum_{1\le i,j\le n-1} 2v_iv_j\frac{\partial^2}{\partial z_{ij}\partial z_{nn}}, \sum_{1\le j\le n-1} v_j\frac{\partial}{\partial z_{jn}}\right)$$

intertwines $\mathcal{O}(X,\mathcal{L}_\lambda)$ and $\mathcal{O}(Y,\mathcal{W}_\lambda^a)$, where $\lambda\in\mathbb{Z}$, $a\in\mathbb{N}$.

Answers

Theorem D3 : $(SO(n,2), SO(n-1,2))$

The differential operator

$$D_{X \to Y, a} := (I_a C_a^{\lambda - \frac{n-1}{2}})\left(-\Delta_{\mathbb{C}^{n-1}}, \frac{\partial}{\partial z_n}\right)$$

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It is a holomorphic version of A. Juhl's operators.

Control of multiplicities of branching rules by the dimension of the space $\mathrm{Sol}_{\mathrm{Jacobi}}(\alpha, \beta,$
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Applications

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Interesting phenomenon occurs even for the tensor products of Verma modules of $\mathfrak{sl}(2, \mathbb{R})$ relying on Kummer connection formulas.

Applications in conformal geometry

Conformal Symmetry breaking operators

F-method in Conformal geometry

Let (X, g) be a pseudo-Riemannian manifold and a Lie group G act conformally on X. I.e. $\exists \Omega \in C^\infty(G \times X, R_{>0})$ such that

$$L_h^* g_{h \cdot x} = \Omega(h,x)^2 g_x \quad \text{for all } h \in G, x \in X$$

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When X is orientable, define $or \colon G \times X \longrightarrow \{\pm 1\}$ by $or(h)(x) = 1$ if $(L_h)_{*x} \colon T_x X \longrightarrow T_{L_h x} X$ is orientation-preserving, and $= -1$ if it is orientation-reversing.

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When X is orientable, define $or: G \times X \longrightarrow \{\pm 1\}$ by $or(h)(x) = 1$ if $(L_h)_{*x}: T_x X \longrightarrow T_{L_h x} X$ is orientation-preserving, and $= -1$ if it is orientation-reversing. Form a family of representations $\varpi_{u,\delta}^{(i)}$ of G with $u \in \mathbb{C}$ and $\delta \in \mathbb{Z}/2\mathbb{Z}$ on the space $\mathcal{E}^i(X)$ of i-forms on X $(0 \le i \le \dim X)$

$$\varpi_{u,\delta}^{(i)}(h)\alpha := or(h)^\delta \Omega(h^{-1}, \cdot)^u L_{h^{-1}}^* \alpha, \quad (h \in G).$$

We also write $\mathcal{E}^i(X)_{u,\delta}$ for these 'conformal representations' on i-forms.

Let Y be an orientable submanifold s.t. g is nondegenerate on the tangent space $T_y Y$ for all $y \in Y$. Then Y is endowed with a pseudo-Riemannian structure $g|_Y$, and we introduce representations $\varpi^{(j)}_{v,\varepsilon}$ on $\mathcal{E}^j(Y)$ $(v \in \mathbb{C}, \varepsilon \in \mathbb{Z}/2\mathbb{Z}, 0 \le j \le \dim Y)$ of the group

$$G' := \{h \in G : h \cdot Y = Y\}$$

which acts conformally on $(Y, g|_Y)$.

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We say that such $\mathcal{D}^{i \to j}$ is a *differential symmetry breaking operator* and denote by $\mathrm{Diff}_{G'}(\mathcal{E}^i(X)_{u,\delta}, \mathcal{E}^j(Y)_{v,\varepsilon})$ the space of differential symmetry breaking operators.

F-method in Conformal geometry

Problem 1

Find a necessary and sufficient condition on 6-tuple $(i, j, u, v, \delta, \varepsilon)$ such that there exist nontrivial differential symmetry breaking operators. More precisely, determine the dimension of $\mathrm{Diff}_{G'}\left(\mathcal{E}^i(X)_{u,\delta}, \mathcal{E}^j(Y)_{v,\varepsilon}\right)$.

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Problem 2

Construct explicitly a basis of $\mathrm{Diff}_{G'}\left(\mathcal{E}^i(X)_{u,\delta}, \mathcal{E}^j(Y)_{v,\varepsilon}\right)$.

Conformal symmetry breaking operators : examples

- If $X = Y$, $G = G'$, and $i = j = 0$, a classical prototype of such operators is the Yamabe operator (conformal Laplacian)

$$\Delta + \frac{n-2}{4(n-1)}\kappa \in \mathrm{Diff}_G(\mathcal{E}^0(X)_{\frac{n}{2}-1,\delta}, \mathcal{E}^0(X)_{\frac{n}{2}+1,\delta}),$$

where n is the dimension of the manifold X, Δ is the Laplace–Beltrami operator, and κ is the scalar curvature.

- Higher order conformally equivariant differential operators : the Paneitz operator (fourth order), and more generally the GJMS operators.

- Analogous differential operators on forms ($i = j$ case) were studied by Branson.

- The exterior derivative d and the codifferential d^* also give examples of conformally covariant operators on forms, for $j = i + 1$ and $i - 1$, respectively, with appropriate choice of $(u, v, \delta, \varepsilon)$.

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Conformal symmetry breaking operators : examples

- Let $Y \neq X$ and $G' \neq G$. Then the restriction operator $\mathrm{Rest}_Y \in \mathrm{Diff}_{G'}\left(\mathcal{E}^i(X)_{u,\delta}, \mathcal{E}^i(Y)_{v,\varepsilon}\right)$ if $u = v$ and $\delta \equiv \varepsilon \equiv 0 \bmod 2$.

- Another example, when Y is of codimension one in X, is $\mathrm{Rest}_Y \circ \iota_{N_Y(X)} \in \mathrm{Diff}_{G'}\left(\mathcal{E}^i(X)_{u,\delta}, \mathcal{E}^{i-1}(Y)_{v,\varepsilon}\right)$ with $v = u + 1$ and $\delta \equiv \varepsilon \equiv 1 \bmod 2$.

- In the model space $(X, Y) = (S^n, S^{n-1})$, the pair (G, G') of conformal groups amounts to $(O(n+1,1), O(n,1))$ modulo center, and Problems 1 and 2 have been recently solved for $i = j = 0$ by Juhl and Kobayashi.

- The case $n = 2$ with $(i, j) = (1, 0)$ gives another interpretation of the Rankin–Cohen brackets using the fact that there are natural homomorphisms

$$\begin{array}{ccc} G_{\mathbb{C}} := SL(2,\mathbb{C}) & \to & O(3,1), \\ \cup & & \cup \\ G_{\mathbb{R}} := SL(2,\mathbb{R}) & \to & O(2,1), \end{array}$$

and appropriate isomorphisms of $G_{\mathbb{C}}$-equivariant line bundles over $\mathbb{P}^1\mathbb{C}$ [Kobayashi, Kubo, P. 2015].

Conformal symmetry breaking operators : examples

- Let $Y \neq X$ and $G' \neq G$. Then the restriction operator $\mathrm{Rest}_Y \in \mathrm{Diff}_{G'}\left(\mathcal{E}^i(X)_{u,\delta}, \mathcal{E}^i(Y)_{v,\varepsilon}\right)$ if $u = v$ and $\delta \equiv \varepsilon \equiv 0 \bmod 2$.
- Another example, when Y is of codimension one in X, is $\mathrm{Rest}_Y \circ \iota_{N_Y(X)} \in \mathrm{Diff}_{G'}\left(\mathcal{E}^i(X)_{u,\delta}, \mathcal{E}^{i-1}(Y)_{v,\varepsilon}\right)$ with $v = u + 1$ and $\delta \equiv \varepsilon \equiv 1 \bmod 2$.
- In the model space $(X, Y) = (S^n, S^{n-1})$, the pair (G, G') of conformal groups amounts to $(O(n+1,1), O(n,1))$ modulo center, and Problems 1 and 2 have been recently solved for $i = j = 0$ by Juhl and Kobayashi.
- The case $n = 2$ with $(i, j) = (1, 0)$ gives another interpretation of the Rankin–Cohen brackets using the fact that there are natural homomorphisms

$$\begin{array}{ccc} G_{\mathbb{C}} := SL(2,\mathbb{C}) & \to & O(3,1), \\ \cup & & \cup \\ G_{\mathbb{R}} := SL(2,\mathbb{R}) & \to & O(2,1), \end{array}$$

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Conformal symmetry breaking operators : main results

(T. Kobayashi, T. Kubo-P. 2016)Solution to Problem 1 for (S^n, S^{n-1})

Let $n \geq 3$. Suppose $0 \leq i \leq n$, $0 \leq j \leq n-1$, $u, v \in \mathbb{C}$, $\delta, \varepsilon \in \mathbb{Z}/2\mathbb{Z}$. Then the following three conditions on 6-tuple $(i, j, u, v, \delta, \varepsilon)$ are equivalent :

(i) $\mathrm{Diff}_{O(n,1)}(\mathcal{E}^i(S^n)_{u,\delta}, \mathcal{E}^j(S^{n-1})_{v,\varepsilon}) \neq \{0\}$,

(ii) $\dim_{\mathbb{C}} \mathrm{Diff}_{O(n,1)}(\mathcal{E}^i(S^n)_{u,\delta}, \mathcal{E}^j(S^{n-1})_{v,\varepsilon}) = 1$,

(iii) One of the following conditions holds.

Case (I). $j = i-2$, $2 \leq i \leq n-1$, $(u, v) = (n-2i, n-2i+3)$, $\delta \equiv \varepsilon \equiv 1 \mod 2$.

Case (I′). $(i, j) = (n, n-2)$, $u \in -n-\mathbb{N}$, $v = 3-n$, $\delta \equiv \varepsilon \equiv u+n+1 \mod 2$.

Case (II). $j = i-1$, $1 \leq i \leq n$, $v-u \in \mathbb{N}_+$, $\delta \equiv \varepsilon \equiv v-u \mod 2$.

Case (III). $j = i$, $0 \leq i \leq n-1$, $v-u \in \mathbb{N}$, $\delta \equiv \varepsilon \equiv v-u \mod 2$.

Case (IV). $j = i+1$, $1 \leq i \leq n-2$, $(u, v) = (0, 0)$, $\delta \equiv \varepsilon \equiv 0 \mod 2$.

Case (IV′). $(i, j) = (0, 1)$, $u \in -\mathbb{N}$, $v = 0$, $\delta \equiv \varepsilon \equiv u \mod 2$.

Case (∗I). $j = n-i+1$, $2 \leq i \leq n-1$, $u = n-2i$, $v = 0$, $\delta \equiv 1$, $\varepsilon \equiv 0 \mod 2$.

Case (∗I′). $(i, j) = (n, 1)$, $u \in -n-\mathbb{N}$, $v = 0$, $\delta \equiv \varepsilon + 1 \equiv u+n+1 \mod 2$.

Case (∗II). $j = n-i$, $1 \leq i \leq n$, $v-n+n-2i \in \mathbb{N}$, $\delta \equiv \varepsilon+1 \equiv v-u+n+1 \mod 2$.

Case (∗III). $j = n-i-1$, $0 \leq i \leq n-1$, $v-u+n-2i-1 \in \mathbb{N}$, $\delta \equiv \varepsilon+1 \equiv v-u+n+1 \mod 2$.

Case (∗IV). $j = n-i-2$, $1 \leq i \leq n-2$, $(u, v) = (0, 2i-n+3)$, $\delta \equiv 0$, $\varepsilon \equiv 1 \mod 2$.

Case (∗IV′). $(i, j) = (0, n-2)$, $u \in -\mathbb{N}$, $v = 3-n$, $\delta \equiv \varepsilon + 1 \equiv u \mod 2$.

Conformal symmetry breaking operators : main results

Recall

$$(I_\ell \widetilde{C}_\ell^\mu)(x,y) := x^{\frac{\ell}{2}} \widetilde{C}_\ell^\mu\left(\frac{y}{\sqrt{x}}\right)$$

and define a family of A. Juhl's scalar-valued differential operators on $\mathbb{R}^n$ of order ℓ

$$\mathcal{D}_\ell^\mu := (I_\ell \widetilde{C}_\ell^\mu)\left(-\Delta_{\mathbb{R}^{n-1}}, \frac{\partial}{\partial x_n}\right).$$

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For instance,

$$\begin{aligned}
\mathcal{D}_0^\mu &= 1, \\
\mathcal{D}_1^\mu &= 2\frac{\partial}{\partial x_n}, \\
\mathcal{D}_2^\mu &= \Delta_{\mathbb{R}^{n-1}} + 2(\mu+1)\frac{\partial^2}{\partial x_n^2}, \\
\mathcal{D}_3^\mu &= 2\Delta_{\mathbb{R}^{n-1}}\frac{\partial}{\partial x_n} + \frac{4}{3}(\mu+2)\frac{\partial^3}{\partial x_n^3}, \cdots
\end{aligned}$$

Conformal symmetry breaking operators : main results

Thus $\mathcal{D}_{u,a}^{i\to i-1}: \mathcal{E}^i(\mathbb{R}^n) \to \mathcal{E}^{i-1}(\mathbb{R}^{n-1})$ with $u \in \mathbb{C}, a \in \mathbb{N}$ is given by

$$\mathcal{D}_{u,a}^{i\to i-1} := \mathrm{Rest}_{x_n=0} \circ \left(-\mathcal{D}_{a-2}^{\mu+1} d_{\mathbb{R}^n} d_{\mathbb{R}^n}^* \iota_{\frac{\partial}{\partial x_n}} - \gamma(\mu,a)\mathcal{D}_{a-1}^{\mu+1} d_{\mathbb{R}^n}^* + \frac{1}{2}(u+2i-n)\mathcal{D}_a^{\mu}\iota_{\frac{\partial}{\partial x_n}} \right)$$
$$= \mathrm{Rest}_{x_n=0} \circ \left(-\mathcal{D}_{a-2}^{\mu+1} d_{\mathbb{R}^n}^* \iota_{\frac{\partial}{\partial x_n}} d_{\mathbb{R}^n} + \frac{1}{2}(u+2i-n+a)\mathcal{D}_a^{\mu}\iota_{\frac{\partial}{\partial x_n}} \right)$$
$$- \gamma(\mu - \frac{1}{2}, a) d_{\mathbb{R}^{n-1}}^* \circ \mathrm{Rest}_{x_n=0} \circ \mathcal{D}_{a-1}^{\mu},$$

where $\gamma(\mu,a) := \frac{\Gamma(\mu+1+[\frac{a}{2}])}{\Gamma(\mu+[\frac{a+1}{2}])} = \begin{cases} [l]1 & \text{if} \quad a \text{ is odd,} \\ \mu + \frac{a}{2} & \text{if} \quad a \text{ is even.} \end{cases}$

Relaies on a finite hierarchy of Fuchsian ODEs coming from the analysis of matrix-valued diff. operators of degree 2

$$\widehat{d\pi_{\sigma,\lambda}}(Y) = S + V$$

where V is a scalar-valued diff.op. of second order depending only on λ and V a matrix-valued diff.op. of first order.

Examples :

$$\mathcal{D}_{u,a}^{1\to 0} = \mathrm{Rest}_{x_n=0}\circ\left(-\gamma(u-\frac{n-3}{2},a)\mathcal{D}_{a-1}^{u-\frac{n-5}{2}}d_{\mathbb{R}^n}^{*}+\frac{1}{2}(u+2-n)\mathcal{D}_{a}^{u-\frac{n-3}{2}}\iota_{\frac{\partial}{\partial x_n}}\right),$$

$$\mathcal{D}_{u,a}^{n\to n-1} = \frac{1}{2}(u+n+a)\mathrm{Rest}_{x_n=0}\circ\mathcal{D}_{a}^{u+\frac{n+1}{2}}\iota_{\frac{\partial}{\partial x_n}},$$

$$\mathcal{D}_{u,0}^{i\to i-1} = \frac{1}{2}(u+2i-n)\mathrm{Rest}_{x_n=0}\circ\iota_{\frac{\partial}{\partial x_n}},$$

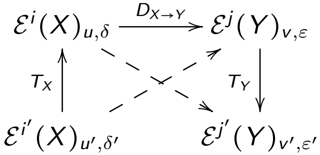
$$\mathcal{D}_{u,1}^{i\to i-1} = \mathrm{Rest}_{x_n=0}\circ\left(-d_{\mathbb{R}^n}^{*}+(u+2i-n)\frac{\partial}{\partial x_n}\iota_{\frac{\partial}{\partial x_n}}\right),$$

$$\mathcal{D}_{u,2}^{i\to i-1} = \mathrm{Rest}_{x_n=0}\circ D,$$

where $D=\left(-d_{\mathbb{R}^n}d_{\mathbb{R}^n}^{*}+\frac{1}{2}(u+2i-n)\left(\Delta_{\mathbb{R}^{n-1}}+(n+2i+5)\frac{\partial^2}{\partial x_n^2}\right)\right)\iota_{\frac{\partial}{\partial x_n}}-(2u-n+2i+3)\frac{\partial}{\partial x_n}d_{\mathbb{R}^n}^{*}$.

Conformal symmetry breaking operators : factorization identities

It may happen :



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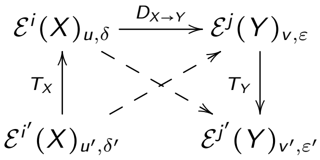

$$(1)\quad \mathcal{D}^{i\to i-1}_{u+2\ell,a}\circ \mathcal{T}^{(i)}_{2\ell} = -\left(\frac{n}{2}-i-\ell\right)K_{\ell,a}\mathcal{D}^{i\to i-1}_{u,a+2\ell}\quad \text{if } i\neq 0.$$

$$(2)\quad \mathcal{D}^{i\to i}_{u+2\ell,a}\circ \mathcal{T}^{(i)}_{2\ell} = -\left(\frac{n}{2}-i+\ell\right)K_{\ell,a}\mathcal{D}^{i\to i}_{u,a+2\ell}\quad \text{if } i\neq n.$$

with $K_{\ell,a} := \prod_{k=1}^{\ell}\left(\left[\frac{a}{2}\right]+k\right)$.

Conformal symmetry breaking operators : factorization identities

In some specific cases

$$
\begin{array}{llll}
(1) & \mathcal{D}_{0,a}^{i+1\to i} \circ d = \gamma(i+1-\frac{n}{2}, a)\mathcal{D}_{0,a+1}^{i\to i}, & 0 \le i \le n-1, & \delta \equiv a+1 \bmod 2. \\
(2) & \mathcal{D}_{0,a}^{i+1\to i+1} \circ d = 0, & 0 \le i \le n-1, & \delta \equiv 0 \bmod 2. \\
(3) & \mathcal{D}_{n-2i+2,a}^{i-1\to i-1} \circ d^* = -\gamma(-i+1+\frac{n}{2}, a)\mathcal{D}_{n-2i,a+1}^{i\to i-1}, & 1 \le i \le n, & \delta \equiv a \bmod 2. \\
(4) & \mathcal{D}_{n-2i+2,a}^{i-1\to i-2} \circ d^* = 0, & 2 \le i \le n, & \delta \equiv 1 \bmod 2.
\end{array}
$$

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Thank you !