Convex and coherent risk measures

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Abstract: We discuss the quantification of financial risk in terms of monetary risk measures. Special emphasis is on dual representations of convex risk measures, relations to expected utility and other valuation concepts, conditioning, and consistency in discrete time.

1 Introduction

Quantifying the risk of the uncertainty in the future value of a portfolio is one of the key tasks of risk management. This quantification is usually achieved by modeling the uncertain payoff as a random variable, to which then a certain functional is applied. Such functionals are usually called *risk measures.* The corresponding industry standard, Value at Risk, is often criticized for encouraging the accummulation of shortfall risk in particular scenarios. This deficiency has lead to a search for more appropriate alternatives. The first step of this search consists in specifying certain desirable axioms for risk measures. In a second step, one then tries to characterize those risk measures that satisfy these axioms and to identify suitable examples. In Section 2, we first provide the various axiom sets for *monetary, convex,* and *coherent* risk measures. Section 3 briefly discusses the representation of monetary risk measures in terms of their *acceptance sets.* The general *dual representation* for convex and coherent risk measures is given in Section 4. Various examples are provided in Section 5. In many situations, it is reasonable to assume that a risk measure depends on the randomness of the portfolio value only through its probability law. Such risk measures are usually called *law-invariant.* They are discussed in Section 6. The final Section 7 analyzes various notions of *dynamic consistency* that naturally arise in a multi-period setting.

2 Monetary, convex, and coherent risk measures

The uncertainty in the future value of a portfolio is usually described by a function $X : \Omega \to \mathbb{R}$, where Ω is a fixed set of scenarios. For instance, X can be the (discounted) value of the portfolio or the sum of its P&L and some economic capital. The goal is to determine a number $\rho(X)$ that quantifies the risk and can serve as a capital requirement, i.e., as the minimal amount of capital which, if added to the position and invested in a risk-free manner, makes the position acceptable. The following axiomatic approach to such risk measures was initiated in the coherent case by [1] and later extended to the class of convex risk measures in [33, 26, 30]. In the sequel, $\mathcal{X}$ denotes a given linear space of functions $X : \Omega \to \mathbb{R}$ containing the constants.

Definition 2.1. A mapping $\rho : \mathcal{X} \to \mathbb{R} \cup \{+\infty\}$ is called a *monetary risk measure* if $\rho(0)$ is finite and if ρ satisfies the following conditions for all $X, Y \in \mathcal{X}$.

- *Monotonicity:* If $X \le Y$, then $\rho(X) \ge \rho(Y)$.
- *Cash invariance:* If $m \in \mathbb{R}$, then $\rho(X + m) = \rho(X) - m$.

The financial meaning of monotonicity is clear: The downside risk of a position is reduced if the payoff profile is increased. Cash invariance is also called translation property; in the normalized case $\rho(0) = 0$ and $\rho(1) = -1$ it is equivalent to *cash additivity*, i.e., $\rho(X + m) = \rho(X) + \rho(m)$. This is motivated by the interpretation of $\rho(X)$ as a capital requirement, i.e., $\rho(X)$ is the amount which should be raised in order to make X acceptable from the point of view of a supervising agency. Thus, if the risk-free amount m is appropriately added to the position or to the economic capital, then the capital requirement is reduced by the same amount. Note that we work with discounted quantities; cf. [22] for a discussion of forward risk measures and interest rate ambiguity.

Definition 2.2. A monetary risk measure ρ is called a *convex risk measure* if it satisfies

- *Convexity:* $\rho(\lambda X + (1-\lambda)Y) \le \lambda\rho(X) + (1-\lambda)\rho(Y)$, for $0 \le \lambda \le 1$.

Consider the collection of possible future outcomes that can be generated with the resources available to an investor: one investment strategy leads to X, while a second strategy leads to Y. If one *diversifies*, spending only the fraction λ of the resources on the first possibility and using the remaining part for the second alternative, one obtains $\lambda X + (1-\lambda)Y$. Thus, the axiom of convexity gives a precise meaning to the idea that diversification should not increase the risk. This idea becomes even clearer when we note that, for a monetary risk measure, convexity is in fact equivalent to the weaker requirement of

- *Quasi Convexity:* $\rho(\lambda X + (1-\lambda)Y) \le \max\big(\rho(X), \rho(Y)\big)$, for $0 \le \lambda \le 1$.

Definition 2.3. A convex measure of risk ρ is called a *coherent risk measure* if it satisfies

- *Positive Homogeneity:* If $\lambda \ge 0$, then $\rho(\lambda X) = \lambda\rho(X)$.

Under the assumption of positive homogeneity, the convexity of a monetary risk measure is equivalent to

- *Subadditivity:* $\rho(X + Y) \le \rho(X) + \rho(Y)$.

This property allows to decentralize the task of managing the risk arising from a collection of different positions: If separate risk limits are given to different "desks", then the risk of the aggregate position is bounded by the sum of the individual risk limits.

Value at Risk at level $\alpha \in]0,1[$, defined for random variables X on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ by

$$V@R_\alpha(X) = \inf\{m \in \mathbb{R} \,|\, \mathbb{P}[\,X + m < 0\,] \leq \alpha\},$$

is a monetary risk measure that is positively homogeneous but not subadditive and hence not convex (see eqf15/004). *Average Value at Risk* at level $\lambda \in]0,1]$,

$$AV@R_\lambda = \frac{1}{\lambda}\int_0^\lambda V@R_\alpha(X)\,d\alpha, \tag{2.1}$$

also called *Conditional Value at Risk*, *Expected Shortfall*, or *Tail Value at Risk* (see eqf15/005), is a coherent risk measure. Other examples are discussed in Section 5.

It is sometimes convenient to reverse signs and to put emphasis on the *utility* of a position rather than on its risk. Thus, if ρ is a convex risk measure, then $\phi(X) := -\rho(X)$ is called a *concave monetary utility functional*. If ρ is coherent then ϕ is called a *coherent monetary utility functional*.

3 Acceptance sets

A monetary measure of risk ρ induces the set

$$\mathcal{A}_\rho := \{\, X \in \mathcal{X} \mid \rho(X) \leq 0 \,\}$$

of positions which are acceptable in the sense that they do not require additional capital. The set $\mathcal{A}_\rho$ is called the *acceptance set* of ρ. One can show that ρ is a convex risk measure if and only if $\mathcal{A}_\rho$ is a convex set and that ρ is positively homogeneous if and only if $\mathcal{A}_\rho$ is a cone. In particular, ρ is coherent if and only if $\mathcal{A}_\rho$ is a convex cone. The acceptance set completely determines ρ, because

$$\rho(X) = \inf\{\, m \in \mathbb{R} \mid m + X \in \mathcal{A}_\rho \,\}. \tag{3.1}$$

Moreover, $\mathcal{A} := \mathcal{A}_\rho$ satisfies the following properties.

$$\mathcal{A} \cap \mathbb{R} \neq \emptyset, \tag{3.2}$$

$$\inf\{\, m \in \mathbb{R} \mid X + m \in \mathcal{A} \,\} > -\infty \text{ for all } X \in \mathcal{X}, \tag{3.3}$$

$$X \in \mathcal{A},\; Y \in \mathcal{X}, Y \geq X \quad \Longrightarrow \quad Y \in \mathcal{A}. \tag{3.4}$$

Conversely, one can take a given class $\mathcal{A} \subset \mathcal{X}$ of acceptable positions as the primary object. For a position $X \in \mathcal{X}$, we can then define

$$\rho_{\mathcal{A}}(X) := \inf\{\, m \in \mathbb{R} \mid m + X \in \mathcal{A} \,\}. \tag{3.5}$$

If $\mathcal{A}$ satisfies the properties (3.2)–(3.4), then $\rho_{\mathcal{A}}$ is a monetary risk measure. If $\mathcal{A}$ is convex, then so is $\rho_{\mathcal{A}}$. If $\mathcal{A}$ is a cone, then $\rho_{\mathcal{A}}$ is positively homogeneous. Note that, with this notation, (3.1) takes the form $\rho_{\mathcal{A}_\rho} = \rho$. The validity of the analogous identity $\mathcal{A}_{\rho_{\mathcal{A}}} = \mathcal{A}$ requires that $\mathcal{A}$ satisfies a certain closure property.

4 Dual representation

Suppose now that $\mathcal{X}$ consists of measurable functions on $(\Omega, \mathcal{F})$. A *dual representation* of a convex risk measure ρ has the form

$$\rho(X) = \sup_{Q \in \mathcal{M}} \big(E_Q[\,-X\,] - \alpha(Q)\big). \tag{4.1}$$

Here $\mathcal{M}$ is a set of probability measures on $(\Omega, \mathcal{F})$ such that $E_Q[\,X\,]$ is well-defined for all $Q \in \mathcal{M}$ and $X \in \mathcal{X}$. The functional $\alpha : \mathcal{M} \to \mathbb{R} \cup \{+\infty\}$ is called *penalty function*.

The elements of $\mathcal{M}$ can be interpreted as possible probabilistic models, which are taken more or less seriously according to the size of the penalty $\alpha(Q)$. Thus, the value $\rho(X)$ is computed as the worst-case expectation taken over all models $Q \in \mathcal{M}$ and penalized by $\alpha(Q)$; see [8, 28, 42].

In the dual representation theory of convex risk measures one aims at deriving a representation (4.1) in a systematic manner. The general idea is to apply convex duality. For every $Q \in \mathcal{M}$, we define the *minimal penalty function* of ρ by

$$\alpha_\rho(Q) := \sup_{X \in \mathcal{X}} \big(E_Q[\,-X\,] - \rho(X)\big) = \sup_{X \in \mathcal{A}_\rho} E_Q[\,-X\,].$$

With additional assumptions on the structure of $\mathcal{X}$ and on continuity properties of ρ it is often possible to derive the representation

$$\rho(X) = \sup_{Q \in \mathcal{M}} \big(E_Q[\,-X\,] - \alpha_\rho(Q)\big) \tag{4.2}$$

via Fenchel-Legendre duality. In this case, ρ is coherent if and only if α_ρ takes only the values 0 and $+\infty$, and so

$$\rho(X) = \sup_{Q \in \mathcal{Q}_\rho} E_Q[\,-X\,], \tag{4.3}$$

where $\mathcal{Q}_\rho$ consists of all $Q \in \mathcal{M}$ with $\alpha_\rho(Q) = 0$. We now discuss some situations in which representations (4.2) can be obtained. In general, however, it may be necessary to consider extended sets $\mathcal{M}$ that also contain, e.g., finitely additive set functions. Dual representation theory goes back to [34, 32, 1, 18, 33, 26, 30].

First, let $\mathcal{X}$ be the space of all bounded measurable functions on $(\Omega, \mathcal{F})$. Then every convex risk measure ρ takes only finite values and is Lipschitz continuous with respect to the supremum norm. For $\mathcal{M}$ we can take the set of all probability measures on $(\Omega, \mathcal{F})$. The validity of the dual representation (4.1) implies that ρ is *continuous from above* in the sense that

$$X_n \searrow X \quad \Longrightarrow \quad \rho(X_n) \nearrow \rho(X). \tag{4.4}$$

On the other hand, the condition of *continuity from below*,

$$X_n \nearrow X \quad \Longrightarrow \quad \rho(X_n) \searrow \rho(X), \tag{4.5}$$

is equivalent to the validity of the *strong representation*

$$\rho(X) = \max_{Q \in \mathcal{M}} \big(E_Q[\,-X\,] - \alpha_\rho(Q)\big) \tag{4.6}$$

in which for every $X \in \mathcal{X}$ the maximum is attained by some $Q \in \mathcal{M}$. In particular, continuity from below is stronger than continuity from above. Continuity from above is equivalent to the so-called *Fatou property*,

$$\liminf_{n \uparrow \infty} \rho(X_n) \geq \rho(X) \text{ for any bounded sequence } (X_n) \text{ converging pointwise to } X. \tag{4.7}$$

Continuity from below is equivalent to the stronger *Lebesgue property*,

$$\lim_{n\uparrow\infty} \rho(X_n) = \rho(X) \text{ for any bounded sequence } (X_n) \text{ converging pointwise to } X. \tag{4.8}$$

Next, we fix a reference probability measure $\mathbb{P}$ on $(\Omega, \mathcal{F})$ and consider the case in which $\mathcal{X} = L^p := L^p(\Omega, \mathcal{F}, \mathbb{P})$ for some $p \in [1, \infty]$. This choice implicitly requires that $\rho(X) = \rho(\widetilde{X})$ whenever $X = \widetilde{X}$ $\mathbb{P}$-almost surely. For $\mathcal{M}$ we take the set of all probability measures that are absolutely continuous with respect to $\mathbb{P}$ and whose density belongs to L^q, where $q = p/(p-1)$ is the dual exponent.

The space $\mathcal{X} = L^\infty$ can be regarded as a subset of the space of all bounded measurable functions, and so all corresponding results carry over. In addition, continuity from above (or, equivalently, the Fatou property) of ρ is now even equivalent to a dual representation (4.2) in terms of probability measures.

For a convex risk measure ρ on $\mathcal{X} = L^p$ with $1 \le p < \infty$, the existence of a dual representation (4.2) is equivalent to the lower semicontinuity of ρ with respect to the standard L^p-norm. If ρ takes only finite values, then it is even Lipschitz continuous and admits a strong representation (4.6). Here we assume for simplicity that $(\Omega, \mathcal{F}, \mathbb{P})$ is atomless and L^2 is separable.

For the discussion of the dual representation of convex risk measures on spaces of bounded measurable functions we refer to [28, 37, 38]. For L^p spaces see [36, 23] and the references therein. Representation theory on Orlicz spaces is considered in [9].

5 Examples and applications

In this section we take $\mathcal{X} = L^\infty(\Omega, \mathcal{F}, \mathbb{P})$. The class of convex risk measures comprises many of the common valuation methods in finance and economics. The risk-neutral expectation in a nice arbitrage-free market model, for instance, clearly corresponds to a coherent risk measure. If the market model is incomplete, then the cost of superheding a position $X \in \mathcal{X}$ is given by the coherent risk measure

$$\sup_{Q \in \mathcal{P}} E_Q[\, -X \,],$$

where $\mathcal{P}$ is the set of equivalent local martingale measures (see eqf04/012). If one imposes additional convex trading constraints, the cost of superheding is a convex risk measure whose representation (4.1) is explicitly known; see [24, 26, 28].

Let us know consider the case in which valuation of positions $X \in \mathcal{X}$ is based on the expected utility $\mathbb{E}[\, u(X) \,]$ for a concave and strictly increasing function $u : \mathbb{R} \to \mathbb{R}$. Then a position can be called acceptable if $E_Q[\, u(X) \,]$ is bounded from below by $u(c)$ for a given threshold c. The set

$$\mathcal{A} := \{\, X \in \mathcal{X} \mid E_Q[\, u(X) \,] \ge u(c) \,\}.$$

is a valid and convex acceptance set. Hence, $\rho_{\mathcal{A}}$, defined via (3.5), is a convex risk measure called *utility-based shortfall risk measure*. It is continuous from below and admits the strong representation (4.6) with minimal penalty function

$$\alpha_\rho(Q) = \inf_{\lambda > 0} \frac{1}{\lambda}\Big(\mathbb{E}\Big[\, \widetilde{u}\Big(\lambda \frac{dQ}{d\mathbb{P}}\Big) \,\Big] - u(c) \Big),$$

where $\widetilde{u}(y) = \sup_x (u(x) - xy)$ denotes the convex conjugate function of u; see [26] or [28]. In the CARA utility case with $u(x) = -e^{-\theta x}$ for some $\theta > 0$, we obtain for $c = 0$ the *entropic risk measure*,

$$\rho^{\text{ent}}(X) = \frac{1}{\theta} \log \mathbb{E}[\, e^{-\theta X} \,]. \tag{5.1}$$

Its minimal penalty function is given by $\alpha_\rho(Q) = \frac{1}{\theta}H(Q|\mathbb{P})$, where

$$H(Q|\mathbb{P}) = \mathbb{E}\Big[\,\frac{dQ}{d\mathbb{P}}\log\frac{dQ}{d\mathbb{P}}\,\Big]$$

is the relative entropy of $Q \ll \mathbb{P}$; see [28]. For the role of entropic risk measures in problems of risk transfer see [3].

To introduce another closely related class of concave monetary utility functionals, let $g : [0,\infty[\to \mathbb{R}\cup\{+\infty\}$ be a lower semicontinuous convex function satisfying $g(1) < \infty$ and the superlinear growth condition $g(x)/x \to +\infty$ as $x \uparrow \infty$. Associated to it is the *g-divergence*

$$I_g(Q|\mathbb{P}) := \mathbb{E}\Big[\,g\Big(\frac{dQ}{d\mathbb{P}}\Big)\,\Big], \qquad Q \ll \mathbb{P}, \tag{5.2}$$

as introduced in [14, 15]. The g-divergence $I_g(Q|\mathbb{P})$ can be interpreted as a statistical distance between the hypothetical model Q and the reference measure $\mathbb{P}$, so that $\gamma_g(Q) := I_g(Q|\mathbb{P})$ is a natural choice for a penalty function. The risk measure

$$\rho_g(X) := \sup_{Q\ll\mathbb{P}}\big(E_Q[\,-X\,] - I_g(Q|\mathbb{P})\big), \tag{5.3}$$

corresponding to such a divergence penalty, is continuous from below, and $I_g(\cdot|\mathbb{P})$ is its minimal penalty function. The corresponding concave utility functional, $\phi_g = -\rho_g$, was called *optimized certainty equivalent* in [5]. This name stems from the variational identity

$$\phi_g(X) = \sup_{z\in\mathbb{R}}\big(\mathbb{E}[\,u(X-z)\,] + z\big), \qquad X \in L^\infty, \tag{5.4}$$

where $u(x) = \inf_{z>0}(xz + g(z))$ is the concave conjugate function of g; see [4, 47]. In [13], the name *divergence utility* is used. Note that the particular choice $g(x) = x\log x$ corresponds to the relative entropy, $I_g(Q|\mathbb{P}) = H(Q|\mathbb{P})$, and so ρ_g coincides with the entropic risk measure. Another important example is provided by taking $g(x) = 0$ for $x \le \lambda^{-1}$ and $g(x) = \infty$ otherwise, so that the corresponding coherent risk measure is given by Average Value at Risk at level λ:

$$AV@R_\lambda(X) = \inf_{Q\in\mathcal{Q}_\lambda} E_Q[\,X\,] \qquad \text{for} \qquad \mathcal{Q}_\lambda := \Big\{Q \ll \mathbb{P}\,\Big|\,\frac{dQ}{d\mathbb{P}} \le \frac{1}{\lambda}\Big\}; \tag{5.5}$$

see (2.1) and eqf15/005. In this case, we have $u(x) = 0 \wedge x/\lambda$ and hence get the classical duality formula

$$AV@R_\lambda(X) = \frac{1}{\lambda}\inf_{z\in\mathbb{R}}\big(\mathbb{E}[\,(z-X)^+\,] - \lambda z\big) \tag{5.6}$$

as a special case of (5.4). The mixtures of Average Value at Risk at various levels λ are called *spectral risk measures*. They are again coherent risk measures and discussed in more detail in eqf15/007.

Many of these risk measures can be extended in a straightforward manner to spaces of unbounded random variables (see [23] for a systematic study of such extensions). For Gaussian random variables X, Value at Risk, Average Value at Risk, and the spectral risk measures all take the form

$$\rho(X) = \mathbb{E}[\,-X\,] + c\cdot\sigma(X),$$

with different constants c; for the entropic risk measure, $\sigma(X)$ is replaced by the variance $\sigma^2(X)$.

Model uncertainty is another situation in which it is natural to consider risk measures, due to the interpretation of the measures Q in the dual representation (4.1) as suitably penalized probabilistic models. This idea already appears in robust statistics [34]. More recently, coherent and convex risk measures were applied in obtaining numerical representations of investors who are averse against both risk and model uncertainty [32, 27, 43, 29] or to define *measures of model uncertainty* [16].

In a financial market model, it makes sense to combine risk measurement with dynamic or static hedges. For instance, measuring the residual risk of a position after hedging by a convex risk measure ρ is equivalent to using the convex risk measure that arises as the *inf-convolution* of ρ and the superhedging risk measure, defined at the beginning of this section; cf. [26, 3, 28, 46] and the references therein.

6 Law-invariant risk measures

Here we discuss those convex risk measures ρ on $\mathcal{X} = L^\infty(\Omega, \mathcal{F}, \mathbb{P})$ that satisfy $\rho(X) = \rho(\widetilde{X})$ for random variables $X, \widetilde{X} \in \mathcal{X}$ that have the same law under $\mathbb{P}$. These risk measures are usually called *law-invariant*. Examples from the preceding section are Average Value at Risk, the spectral risk measures, the utility-based shortfall risk measures, and the optimized certainty equivalents. Under mild conditions on the underlying probility space, every law-invariant convex risk measure ρ can be represented in the form

$$\rho(X) = \sup_{\mu} \Big(\int_{(0,1]} AV@R_\lambda(X)\, \mu(d\lambda) - \beta(\mu) \Big), \tag{6.1}$$

where the supremum is taken over all Borel probability measures μ on $]0, 1]$ and $\beta(\mu)$ is a penalty for μ. Under the additional assumption of continuity from above, this representation was obtained in the coherent case by [41] and later extended by [39, 17, 28, 31]. More recently, it was shown in [35] that the condition of continuity from above can actually be dropped.

7 Conditional convex risk measures and time-consistency

A risk measure should take into account the available information, and it should do so in a consistent manner as new information comes in. Here we limit the discussion to discrete time and fix a filtration $(\mathcal{F}_t)_{t=0,1,\dots}$ on $(\Omega, \mathcal{F}, \mathbb{P})$; for continuous time and the connection to backward stochastic differential equations (BSDE) see [20] and the references therein. A *conditional convex risk measure* at time t is now defined as a map

$$\rho_t \,:\, L^\infty \to\, L_t^\infty := L^\infty(\Omega, \mathcal{F}_t, \mathbb{P})$$

which satisfies the obvious conditional versions of monotonicity, cash-invariance, and convexity where the constants m and λ are replaced by functions in L_t^∞. The associated *acceptance set*

$$\mathcal{A}_t := \{X \in L^\infty \mid \rho_t(X) \le 0\}$$

is conditionally convex (i.e., $\alpha X + (1-\alpha) Y \in \mathcal{A}_t$ for X, $Y \in \mathcal{A}_t$ and $\mathcal{F}_{t-1}$-measurable α with $0 \le \alpha \le 1$) and it determines ρ_t via

$$\rho_t(X) = \operatorname{ess\,inf} \{Y \in L_t^\infty \mid X + Y \in \mathcal{A}_t\}.$$

The *Fatou property* is now equivalent to a *dual representation* of the form

$$\rho_t(X) = \operatorname*{ess\,sup}_{Q \in \mathcal{M}} \left(E_Q[\, -X \mid \mathcal{F}_t \,] - \alpha_t(Q)\right),$$

where the conditional penalty function α_t is given by

$$\alpha_t(Q) = \operatorname*{ess\,sup}_{X \in \mathcal{A}_t} \; E_Q[\, -X \mid \mathcal{F}_t \,].$$

The inequality $\geq$ immediately follows from the definition of α_t, and the converse inequality is obtained by using the dual representation of the unconditional convex risk measure $\rho(X) := \mathbb{E}[\, \rho_t(X) \,]$; cf. [45, 21, 6, 11, 25].

For the conditional entropic risk measure,

$$\rho_t^{\text{ent}}(X) = \frac{1}{\theta} \log \mathbb{E}[\, e^{-\theta X} \mid \mathcal{F}_t \,],$$

the dual representation holds with

$$\alpha_t(Q) = \frac{1}{\theta} \widehat{H}_t(Q|P),$$

where

$$\widehat{H}_t(Q|P) := \mathbb{E}\Big[\, \frac{Z}{Z_t} \log \frac{Z}{Z_t} \,\Big|\, \mathcal{F}_t \Big] I_{\{Z_t > 0\}}$$

denotes the *conditional entropy* of $Q \in \mathcal{M}$ with respect to P, defined in terms of the densities $Z = dQ/dP$ and $Z_t = dQ/dP|_{\mathcal{F}_t}$.

In our dynamic setting the key question is how the conditional risk assessments of a financial position at different times are connected to each other.

Definition. A *dynamic risk measure* given by a sequence of conditional convex risk measures $(\rho_t)_{t=0,1,\ldots}$ is called *time-consistent* if

$$\rho_{t+1}(X) \leq \rho_{t+1}(Y) \quad \Longrightarrow \quad \rho_t(X) \leq \rho_t(Y),$$

and this is equivalent to *recursiveness*:

$$\rho_t = \rho_t(-\rho_{t+1}), \text{ for } t = 0, 1, \ldots$$

In order to characterize time-consistency in terms of acceptance sets and penalty functions we define the "myopic" acceptance sets

$$\mathcal{A}_{t,t+1} := \{\, X \in L^\infty_{t+1} \mid \rho_t(X) \leq 0\}$$

and the corresponding "myopic" penalty functions

$$\alpha_{t,t+1}(Q) := \operatorname*{ess\,sup}_{X \in \mathcal{A}_{t,t+1}} \; E_Q[\, -X \mid \mathcal{F}_t \,].$$

We also assume that the class $\mathcal{Q}^*$ of all equivalent probability measures Q with $\alpha_0(Q) < \infty$ is not empty. Then time-consistency is equivalent to each of the following conditions:

- $\mathcal{A}_t = \mathcal{A}_{t,t+1} + \mathcal{A}_{t+1} \quad$ for $t = 0, 1, \ldots$
- For any $Q \in \mathcal{M}$,

$$\alpha_t(Q) = \alpha_{t,t+1}(Q) + E_Q[\, \alpha_{t+1} \mid \mathcal{F}_t \,] \quad \text{for } t = 0, 1, \ldots$$

- For any $Q \in \mathcal{Q}^*$ and any $X \in L^\infty$, the process

$$\rho_t(X) + \alpha_t(Q), \quad t = 0, 1, \ldots$$

  is a Q-supermartingale;

cf. [2, 19, 7, 11, 25, 12]. Moreover, each condition implies that the dynamic risk measure admits a robust representation in terms of the set $\mathcal{Q}^*$, i.e.,

$$\rho_t(X) = \operatorname*{ess\,sup}_{Q \in \mathcal{Q}^*} \left(E_Q[\, -X \,|\, \mathcal{F}_t \,] - \alpha_t(Q) \right) \tag{1}$$

for all $X \in L^\infty$ and all $t \geq 0$; cf. [25].

The entropic dynamic risk measure is time-consistent as long as the parameter θ remains constant. On the other hand, time-consistency fails for the dynamic risk measure defined by conditional Average Value at Risk. Under the assumption of *law invariance*, the entropic case is in fact the only time-consistent example, if we include the limiting cases $\theta = 0$ and $\theta = \infty$ corresponding to the conditional expected loss under $\mathbb{P}$ and the *conditional worst-case risk measure*,

$$\rho_t(X) = \operatorname{ess\,inf} \{ Y \in L_t^\infty \,|\, Y \geq -X \},$$

respectively; cf. [40]. This suggest to consider weaker versions of time consistency. For example, the supermartingale property above implies that for each $Q \in \mathcal{Q}^*$ the process $\alpha_t(Q), t = 0, 1, \ldots$, is a Q-supermartingale, and this is equivalent to the weaker requirement

$$\rho_{t+1}(X) \leq 0 \quad \Longrightarrow \quad \rho_t(X) \leq 0,$$

i.e., $\mathcal{A}_t \subseteq \mathcal{A}_{t+1}$ for all $t = 0, 1, \ldots$. In the law invariant case, such weaker notions of consistency may be used for a characterization of utility-based shortfall risk; cf. [48]. The notion of *prudence* introduced in [44] requires

$$X \in \mathcal{A}_t \quad \Longrightarrow \quad -\rho_{t+s}(X) \in \mathcal{A}_t \text{ for all } s \geq 0,$$

and this is characterized by the fact that

$$\rho_t(X) - \sum_{k=0}^{t-1} \alpha_k(Q), \quad t = 0, 1, \ldots$$

is a Q-supermartingale for any $Q \in \mathcal{Q}^*$ and any $X \in L^\infty$.

For an infinite time horizon the supermartingale criteria for time-consistency and for prudence both yield almost sure convergence of the capital requirements $\rho_t(X)$ to an asymptotic capital requirement $\rho_\infty(X)$. We may now ask whether the sequence is *asymptotically safe* in the sense that $\rho_\infty(X) \geq -X$, or even *asymptotically precise* in the sense of $\rho_\infty(X) = -X$; note that asymptotic precision can be viewed as a non-linear analogue of martingale convergence. Criteria in terms of acceptance sets and penalty functions are derived in [25] for the time-consistent case and in [44] for the case of prudence.

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