Consistent Risk Measures and a Non-linear Extension of Backwards Martingale Convergence

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Abstract

We study the behavior of conditional risk measures along decreasing σ-fields. Under a condition of consistency, we prove a non-linear extension of backwards martingale convergence. In particular we show the existence of a limiting conditional risk measure with respect to the tail field, we describe its dual representation in terms of a limiting penalty function, and we show that consistency extends to the tail field. Moreover, we clarify the structure of global risk measures which are consistent with the given sequence of conditional risk measures.

1 Introduction

Consider a filtration $(\mathcal{F}_n)_{n\in\mathbb{Z}}$, indexed by the integers, on some measurable space $(\Omega, \mathcal{F})$. In the forward direction we define the asymptotic σ-field $\mathcal{F}_\infty := \sigma\left(\bigcup_n \mathcal{F}_n\right)$, in the backward direction the tail field $\mathcal{F}_{-\infty} := \bigcap_n \mathcal{F}_n$.

For a given probability measure P and for any bounded measurable function X on $(\Omega, \mathcal{F})$, let us denote by

$$\eta_n(X) := E_P\left[-X|\mathcal{F}_n\right], \qquad n \in \mathbb{Z} \tag{1}$$

the conditional expectation of $-X$ with respect to $\mathcal{F}_n$ under the measure P. Since we are using the minus sign, the functional η_n can be regarded as the special linear case of a conditional convex risk measure, as explained below.

Due to the projectivity of conditional expectations, the sequence $(\eta_n)_{n\in\mathbb{Z}}$ is *consistent* in the sense that

$$\eta_n(-\eta_{n+1}(X)) = \eta_n(X), \qquad n \in \mathbb{Z}. \tag{2}$$

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Martingale convergence forwards and backwards yields the existence of the limits

$$\eta_\infty(X) := \lim_{n\uparrow\infty} \eta_n(X), \quad \eta_{-\infty}(X) := \lim_{n\downarrow-\infty} \eta_n(X)$$

P-a.s. and in $L^1(P)$, and these limits are identified as conditional expectations

$$\eta_\infty(X) = E_P\left[-X|\mathcal{F}_\infty\right], \quad \eta_{-\infty}(X) = E_P\left[-X|\mathcal{F}_{-\infty}\right]$$

with respect to the limiting σ-fields $\mathcal{F}_\infty$ and $\mathcal{F}_{-\infty}$. Again by projectivity, we see that the consistency relation (2) extends to infinity in both directions, that is,

$$\eta_n(-\eta_\infty) = \eta_n \quad \text{and} \quad \eta_{-\infty}(-\eta_n) = \eta_{-\infty} \tag{3}$$

for any $n \in \mathbb{Z}$. Let us summarize these classical facts by saying that the sequence $(\eta_n)_{n\in\mathbb{Z}}$ is *asymptotically precise* in both directions.

In this paper, we study the question whether asymptotic precision extends from the linear case of conditional expectation to the non-linear case of conditional risk measures. For each $n \in \mathbb{Z}$, let ρ_n denote a conditional convex risk measure on $L^\infty(\Omega, \mathcal{F}, P)$ with respect to $\mathcal{F}_n$, and let

$$\mathcal{A}_n := \left\{ X \in L^\infty(\Omega, \mathcal{F}, P) \;\middle|\; \rho_n(X) \le 0\right\}$$

denote the corresponding acceptance set; see, e.g., [19, Chapter 11]. Under an additional continuity assumption, the conditional risk measure ρ_n admits the dual representation

$$\rho_n(X) = \operatorname*{ess\,sup}_{\substack{Q \ll P \\ Q \approx P \text{ on } \mathcal{F}_n}} \left(E_Q\left[-X|\mathcal{F}_n\right] - \alpha_n(Q)\right) \tag{4}$$

with penalty function

$$\alpha_n(Q) = \operatorname*{ess\,sup}_{X\in\mathcal{A}_n} E_Q\left[-X|\mathcal{F}_n\right].$$

In the special coherent case where ρ_n is also positively homogeneous, this reduces to the representation

$$\rho_n(X) = \operatorname*{ess\,sup}_{Q\in\mathcal{Q}_n} E_Q\left[-X|\mathcal{F}_n\right] \tag{5}$$

with a suitable class $\mathcal{Q}_n$ of probability measures Q. Under the additional condition of comonotonicity, the coherent risk measure in (5) can also be regarded as a conditional Choquet integral

$$\rho_n(X) = \int (-X) dC_n,$$

where $C_n(A) := \rho_n(-I_A)$ is a conditional Choquet capacity, in analogy to the discussion in [19, Section 4.7]. Clearly, we recover the conditional expectation

$\eta_n(X)$ in (1) in the simple special case, where the set $\mathcal{Q}_n$ reduces to the single probability measure P.

Let us now assume that the sequence $(\rho_n)_{n\in\mathbb{Z}}$ of conditional risk measures is consistent in the sense of (2), i.e.,

$$\rho_n(-\rho_{n+1}(X)) = \rho_n(X), \qquad n \in \mathbb{Z} \tag{6}$$

for any $X \in L^\infty(\Omega, \mathcal{F}, P)$. Consistency can be characterized in terms of the acceptance sets $(\mathcal{A}_n)_{n\in\mathbb{Z}}$, in terms of the penalty functions $(\alpha_n)_{n\in\mathbb{Z}}$, and also by supermartingale criteria for the joint behavior of (ρ_n) and (α_n); this is recalled in Section 2.2.

In the forward direction, the behavior of the consistent sequence (ρ_n) along the filtration $(\mathcal{F}_n)_{n\geq 0}$ has been studied in [17]. The supermartingale criteria for consistency yield existence of the limit

$$\rho_\infty(X) := \lim_n \rho_n(X).$$

The question is whether ρ_∞ has good properties as a conditional risk measure with respect to $\mathcal{F}_\infty$. In the case $\mathcal{F}_\infty = \mathcal{F}$, asymptotic precision in the forward direction amounts to the condition $\rho_\infty(X) = -X$. However, neither asymptotic precision nor the weaker condition $\rho_\infty(X) \geq -X$ of *asymptotic safety* may hold; see [17, Section 5] for criteria and for counterexamples.

In this paper, we focus on the backward direction, and so it is enough to consider the filtration $(\mathcal{F}_n)_{n\leq 0}$. Under a mild condition on the penalties for our reference measure P, we show in Section 3 that asymptotic precision is indeed satisfied along decreasing σ-fields. More precisely, an application of the supermartingale criteria for consistency yields the existence of the limit

$$\rho_{-\infty}(X) = \lim_{n\downarrow -\infty} \rho_n(X).$$

We then show that the functional $\rho_{-\infty}$ defines a conditional convex risk measure with respect to the tail field $\mathcal{F}_{-\infty}$, that this risk measure is continuous from above, and that its dual representation (4) for $n = -\infty$ is given by the limiting penalty function

$$\alpha_{-\infty}(Q) = \lim_n \alpha_n(Q).$$

Moreover, we show that the consistency condition (6) extends to $-\infty$, that is,

$$\rho_{-\infty}(-\rho_n) = \rho_{-\infty}$$

for any $n \leq 0$, in analogy to (3). In particular, these properties of asymptotic precision in the backward direction hold for a consistent sequence of conditional coherent risk measures, and also for the special case of conditional Choquet integrals.

In the final Section 4 we study the structure of the set $\mathcal{R}$ of all global (unconditional) risk measures ρ on $L^\infty(\Omega, \mathcal{F}, P)$, which are consistent with the given sequence $(\rho_n)_{n\leq 0}$. Under additional continuity conditions, we show that such risk measures are of the form

$$\rho = \hat{\rho}(-\rho_{-\infty}),$$

where $\hat{\rho}$ is a convex risk measure on the tail field; the precise formulation is given in Theorem 5 and Corollary 5.

Our discussion of the behavior of conditional convex risk measures along decreasing σ-fields is motivated by the problem of clarifying the structure of spatial risk measures consistent with a given local specification in a large network. Under a condition of local law-invariance, the local conditional risk measures must be entropic, and then the problem can be solved explicitly, as shown in [15]. Without this condition, the main problem consists in extending the local specification to the tail-field, and this can be done by using the general convergence results of the present paper. The application to spatial risk measures will be discussed in [16].

2 Preliminaries

Throughout this paper we fix a probability space $(\Omega, \mathcal{F}, P)$. We write $L^\infty := L^\infty(\Omega, \mathcal{F}, P)$ and denote by $\mathcal{M}_1(P)$ the set of all probability measures absolutely continuous with respect to P.

In this section we recall some basic facts about conditional convex risk measures and about consistency that will be used later on. For further details see, for example, [13, 4, 17, 5, 8, 1], and [19, Chapter 11].

2.1 Conditional convex risk measures

Let $\mathcal{F}_0$ be a sub-σ-field of $\mathcal{F}$ and write $L_0^\infty := L^\infty(\Omega, \mathcal{F}, P)$.

Definition 1. *A map*

$$\rho_0 \,:\, L^\infty \,\rightarrow\, L_0^\infty$$

is called a conditional convex risk measure with respect to $\mathcal{F}_0$ *if it satisfies the following properties for any* $X, Y \in L^\infty$*:*

- *Conditional cash invariance: For all* $X_0 \in L_0^\infty$*,*

$$\rho_0(X + X_0) = \rho_0(X) - X_0$$

- *Monotonicity:* $X \leq Y \;\Rightarrow\; \rho_0(X) \geq \rho_0(Y)$
- *Conditional convexity: For all* $\lambda \in L_0^\infty$ *such that* $0 \leq \lambda \leq 1$*,*

$$\rho_0(\lambda X + (1-\lambda)Y) \leq \lambda\rho_0(X) + (1-\lambda)\rho_0(Y)$$

- *Normalization:* $\rho_0(0) = 0$*.*

A conditional convex risk measure ρ_0 is called a conditional coherent risk measure *if it has in addition the following property:*

- *Conditional positive homogeneity: For all $\lambda \in L_0^\infty$ such that $\lambda \geq 0$,*

$$\rho_0(\lambda X) = \lambda \rho_0(X).$$

Remark 1. *A conditional convex risk measure ρ_0 is uniquely determined by the associated* acceptance set

$$\mathcal{A}_0 := \left\{ X \in L^\infty \;\middle|\; \rho_0(X) \leq 0 \right\},$$

since

$$\rho_0(X) = \operatorname{ess\,inf} \left\{ Y \in L_0^\infty \;\middle|\; X + Y \in \mathcal{A}_0 \right\}. \tag{7}$$

Thus $\rho_n(X)$ has the financial interpretation of a capital requirement, namely the minimal amount which should be added to the position X to make it acceptable.

Note that $\mathcal{A}_0$ is conditionally convex and solid, and that $\rho_0(0) = 0$ implies $0 \in \mathcal{A}_0$ and $\operatorname{ess\,inf} \left\{ X \in L_0^\infty \;\middle|\; X \in \mathcal{A}_0 \right\} = 0$. Conversely, any set $\mathcal{A}_0$ with these properties defines via (7) a conditional convex risk measure ρ_0.

Under an additional continuity condition, the conditional convex risk measure ρ_0 admits the following dual representation in terms of suitably penalized probability measures $Q \in \mathcal{M}_1(P)$; this is also called the *robust representation* of ρ_0.

For any $Q \in \mathcal{M}_1(P)$ we define

$$\alpha_0(Q) := \operatorname*{ess\,sup}_{X \in \mathcal{A}_0} E_Q\left[-X|\mathcal{F}_0\right]. \tag{8}$$

Q-almost surely, taking the essential supremum under Q. Clearly, $\alpha_0(Q)$ is well defined P-almost surely if Q is equivalent to P on $\mathcal{F}_0$, and in that case (8) can be read as well as an essential supremum under P.

Remark 2.
1. *Since $0 \in \mathcal{A}_0$, we have $\alpha_0(Q) \geq 0$ Q-a.s., and hence P-a.s. if $Q \approx P$ on $\mathcal{F}_0$.*
2. *For any $X \in L^\infty$ we have $X + \rho_0(X) \in \mathcal{A}_0$, and so (8) implies*

$$\rho_0(X) \geq E_Q\left[-X|\mathcal{F}_0\right] - \alpha_0(Q) \quad Q\text{-a.s.} \tag{9}$$

for any $Q \in \mathcal{M}_1(P)$.

With this definition of the penalty function α_0 the following equivalence holds; see [13, 4, 17, 6, 8, 1], and [19].

Theorem 1. *For a conditional convex risk measure ρ_0 with respect to $\mathcal{F}_0$, the following are equivalent:*

1. *ρ_0 has the robust representation*

$$\rho_0(X) = \operatorname*{ess\,sup}_{\substack{Q \in \mathcal{M}_1(P) \\ Q \approx P \; on \, \mathcal{F}_0}} \left(E_Q\left[-X|\mathcal{F}_0\right] - \alpha_0(Q) \right), \quad X \in L^\infty, \tag{10}$$

   where the essential supremum is taken under P.

2. *ρ_0 is continuous from above, i.e.,*

$$X_k \searrow X \quad P\text{-}a.s \quad \Longrightarrow \quad \rho_0(X_k) \nearrow \rho_0(X) \quad P\text{-}a.s$$

   for $X \in L^\infty$ and any sequence $(X_k) \subseteq L^\infty$.

Remark 3. *The penalty function α_0 is* minimal *in the following sense: If the representation* (10) *holds with some function $\tilde{\alpha}_0$, then*

$$\tilde{\alpha}_0(Q) \geq \alpha_0(Q) \quad P\text{-}a.s. \tag{11}$$

for any $Q \in \mathcal{M}_1(P)$ such that $Q \approx P$ on $\mathcal{F}_0$. Indeed, (10) *implies*

$$\tilde{\alpha}_0(Q) \geq E_Q\left[-X|\mathcal{F}_0\right] - \rho_0(X) = E_Q\left[-(X + \rho_0(X))|\mathcal{F}_0\right] \quad P\text{-}a.s.,$$

and hence (11) *in view of* (8), *since $X + \rho_0(X) \in \mathcal{A}_0$.*

Remark 4. *Continuity from above is equivalent to the following condition, also called the* Fatou property*:*

$$\rho_0(X) \leq \liminf_{k\to\infty} \rho_0(X_k)$$

for any uniformly bounded sequence $(X_k) \subset L^\infty$ which converges P-a.s. to some $X \in L^\infty$. We say that ρ_0 has the Lebesgue property*, if the inequality in the preceding condition can be replaced by the equality*

$$\rho_0(X) = \lim_{k\to\infty} \rho_0(X_k).$$

The Lebesgue property holds if and only if ρ_0 is not only continuous from above but also continuous from below*, that is,*

$$X_k \nearrow X \quad P\text{-}a.s \quad \Longrightarrow \quad \rho_0(X_k) \searrow \rho_0(X) \quad P\text{-}a.s..$$

Moreover, it can be shown that the Lebesgue property is equivalent to the condition that the essential supremum in (10) *is actually attained by some measure Q depending on X; for a proof in the unconditional case $\mathcal{F}_0 = \{\Omega, \emptyset\}$ see [12, Theorem 2].*

The proof of Theorem 1 shows that the robust representation in (10) can actually be refined in the sense that we can use a smaller set of probability measures; see, e.g., [17] or [19, Chapter 11].

Corollary 1. *If ρ_0 is continuous from above then we have*

$$\rho_0(X) = \operatorname*{ess\,sup}_{Q \in \mathcal{Q}_0} (E_Q[-X|\mathcal{F}_0] - \alpha_0(Q)), \quad X \in L^\infty,$$

where

$$\mathcal{Q}_0 := \{ Q \in \mathcal{M}_1(P) \mid Q = P \text{ on } \mathcal{F}_0, \ E_Q[\alpha_0(Q)] < \infty \}.$$

Remark 5. *In the special case $\mathcal{F}_0 = \{\Omega, \emptyset\}$, the preceding discussion reduces to standard definitions and basic facts for (unconditional) convex risk measures*

$$\rho : L^\infty \to \mathbb{R}$$

on the Banach space L^∞; see, [2, 3, 10, 18, 20], and also [19, Chapter 4].

2.2 Consistency

Let us now fix two sub-σ-fields $\mathcal{F}_0 \subseteq \mathcal{F}_1$ of $\mathcal{F}$. For $i = 0, 1$, we write $L_i^\infty := L^\infty(\Omega, \mathcal{F}_i, P)$, and we consider a conditional convex risk measure $\rho_i : L^\infty \to L_i^\infty$ with respect to $\mathcal{F}_i$.

Definition 2. *We say that the conditional risk measures ρ_0 and ρ_1 are* consistent *if*

$$\rho_0 = \rho_0(-\rho_1),$$

that is, if $\rho_0(-\rho_1(X)) = \rho_0(X)$ for all $X \in L^\infty$.

From now on we assume that both ρ_0 and ρ_1 are continuous from above. Let $\mathcal{A}_i$ and α_i denote the acceptance set and the minimal penalty function corresponding to ρ_i. Consistency of ρ_0 and ρ_1 can then be characterized in terms of the acceptance sets, in terms of the minimal penalty functions, and in terms of the joint behavior of (ρ_i) and (α_i). To this end, consider the restriction of ρ_0 to the subspace L_1^∞ of L^∞ and denote by

$$\mathcal{A}_{0,1} := \{ X \in L_1^\infty \mid \rho_0(X) \le 0 \ P\text{-a.s.} \}$$

the acceptance set and by

$$\alpha_{0,1}(Q) := \operatorname*{ess\,sup}_{X \in \mathcal{A}_{0,1}} E_Q[-X|\mathcal{F}_0], \qquad Q \in \mathcal{M}_1(P)$$

the minimal penalty function associated to this restriction in analogy to (8). As shown in [11, 17, 5, 8, 9, 6, 1], consistency can now be characterized as follows

Theorem 2. *The following conditions are equivalent:*

1. *ρ_0 and ρ_1 are consistent.*
2. *$\mathcal{A}_0 = \mathcal{A}_{0,1} + \mathcal{A}_1$.*
3. *For any $Q \in \mathcal{M}_1(P)$,*

$$\alpha_0(Q) = \alpha_{0,1}(Q) + E_Q[\, \alpha_1(Q) \,|\, \mathcal{F}_0 \,] \quad Q\text{-a.s.}$$

4. For $X \in L^\infty$ and any $Q \in \mathcal{M}_1(P)$,

$$E_Q[\,\rho_1(X) + \alpha_1(Q)\,|\,\mathcal{F}_0] \leq \rho_0(X) + \alpha_0(Q) \quad Q\text{-a.s..}$$

Remark 6. *All our penalty functions are non-negative, since we have assumed that all our risk measures are normalized. Thus property (3) of Theorem 2 implies that*

$$\alpha_0(Q) \geq E_Q[\alpha_1(Q)|\mathcal{F}_0] \quad Q\text{-a.s. for all } Q \in \mathcal{M}_1(P). \tag{12}$$

In particular, $(\alpha_i(Q))_{i=0,1}$ is a non-negative supermartingale with respect to Q for all $Q \in \mathcal{M}_1(P)$ such that $E_Q[\alpha_0(Q)] < \infty$. Note that the consistency criterion (3) of Theorem 2 provides, in addition to the supermartingale inequality (12)*, a special form of the predictable increasing process in the Doob decomposition of $(\alpha_i)_{i=0,1}$.*

Condition (12) *is equivalent to* weak consistency *of $(\rho_i)_{i=0,1}$, that is, to the condition that*

$$\rho_1(X) \leq 0 \quad \Longrightarrow \quad \rho_0(X) \leq 0$$

for any $X \in L^\infty$; cf. [1, Proposition 8]. Note that weak consistency amounts to the relaxation $\mathcal{A}_1 \subseteq \mathcal{A}_0$ of the consistency criterion (2) in Theorem 2. For other relaxations of the strong notion of consistency in Definition 2 see, for example, [26, 27, 25, 24, 1, 14], and in the law-invariant case [28].

In Section 4 we are going to use the Lebesgue property of conditional risk measures that was introduced in Remark 4, and we will apply the criterion of Proposition 1. This involves the following notion of strong sensitivity; see also [24].

Definition 3. *We call a conditional convex risk measure ρ_0 strongly sensitive with respect to P if*

$$P\,[\rho_0(X) < \rho_0(Y)] > 0$$

whenever $X, Y \in L^\infty$ satisfy $X \geq Y$ P-a.s. and $P\,[X > Y] > 0$.

Proposition 1. *Let ρ_0 and ρ_1 be consistent, and assume that ρ_0 has the Lebesgue property and is strongly sensitive. Then ρ_1 inherits the Lebesgue property and is strongly sensitive.*

Proof. For $X \in L^\infty$ and a uniformly bounded sequence (X_k) in L^∞ such that $X_k \to X$ P-a.s., the Fatou property of ρ_1 yields

$$\rho_1(X) \leq \liminf_k \rho_1(X_k) \leq \limsup_k \rho_1(X_k) \quad P\text{-a.s..} \tag{13}$$

To prove the Lebesgue property of ρ_1, we have to show that

$$\rho_1(X) = \limsup_k \rho_1(X_k) \quad P\text{-a.s..}$$

In view of (13), this will follow from

$$\rho_0(-\rho_1(X)) = \rho_0(-\limsup_k \rho_1(X_k)),$$

due to the strong sensitivity of ρ_0. Indeed, using consistency, monotonicity of ρ_0 applied to (13), and first the Fatou property and then the Lebesgue property of ρ_0, we obtain

$$\begin{aligned}\rho_0(X) = \rho_0(-\rho_1(X)) &\leq \rho_0(-\limsup_k \rho_1(X_k)) \\ &= \rho_0(\liminf_k \rho_1(-X_k)) \leq \liminf_k \rho_0(-\rho_1(X_k)) \\ &= \liminf_k \rho_0(X_k) = \rho_0(X).\end{aligned}$$

To see that ρ_1 is strongly sensitive, take $X, Y \in L^\infty$ such that $X \geq Y$ and $P[X > Y] > 0$. Then we have $P[\rho_1(X) < \rho_1(Y)] > 0$, since $\rho_1(X) = \rho_1(Y)$ P-a.s. would imply

$$\rho_0(X) = \rho_0(-\rho_1(X)) = \rho_0(-\rho_1(Y)) = \rho_0(Y)$$

in contradiction to the strong sensitivity of ρ_0. □

3 Backwards Convergence

From now on we fix a filtration $(\mathcal{F}_n)_{n\leq 0}$ on our probability space $(\Omega, \mathcal{F}, P)$. Thus, the σ-fields $\mathcal{F}_n \subseteq \mathcal{F}$ are decreasing as n decreases to $-\infty$. We denote by

$$\mathcal{F}_{-\infty} := \bigcap_{n\leq 0} \mathcal{F}_n$$

the corresponding *tail field* and write $L^\infty_n = L^\infty(\Omega, \mathcal{F}_n, P)$.

Let $(\rho_n)_{n\leq 0}$ be a sequence of conditional convex risk measures

$$\rho_n : L^\infty \to L^\infty_n.$$

We denote by $\mathcal{A}_n$ the acceptance set of ρ_n, and we assume that each ρ_n is continuous from above. Thus ρ_n admits a dual representation (10) in terms of its minimal penalty function α_n. We also assume that the sequence is *consistent* in the sense that

$$\rho_n(-\rho_{n+1}) = \rho_n \tag{14}$$

for all $n < 0$.

Example 1. *For $\beta \geq 0$ consider the conditional* entropic *risk measures $(\rho_n)_{n\leq 0}$ defined by*

$$\rho_n(X) := \frac{1}{\beta} \log E_P\left[e^{-\beta X} | \mathcal{F}_n\right]; \tag{15}$$

for $\beta = 0$ this is interpreted as the linear case (1)*, that is, as the limiting case of* (15) *as β decreases to* 0*. For $\beta > 0$, the corresponding penalty functions are given by*

$$\alpha_n(Q) = \frac{1}{\beta} H_n(Q|P),$$

where $H_n(Q|P)$ denotes the conditional relative entropy with respect to $\mathcal{F}_n$; see [17] or [19, Chapter 11]. It is easy to check that the sequence $(\rho_n)_{n\le 0}$ is consistent. Note that ρ_n is law-invariant *in the sense that $\rho_n(X)$ only depends on the conditional distribution of X with respect to $\mathcal{F}_n$ under P. Conversely, law-invariance together with consistency implies that the risk measures ρ_n are entropic, if the parameter β is allowed to be tail-measurable with values in $[0,\infty)$; see [15] and also [22]. In this special entropic case, the sequence $(\rho_n)_{n\le 0}$ admits an immediate extension*

$$\rho_{-\infty}(X) = \frac{1}{\beta} \log E_P \left[e^{-\beta X} | \mathcal{F}_{-\infty} \right]$$

to the tail field, and the properties of asymptotic precision are clearly satisfied.

Remark 7. *In general, let $(\tilde{\rho_n})_{n\le 0}$ be any sequence of conditional convex risk measures, not necessarily consistent. Defining recursively*

$$\rho_0 := \tilde{\rho}_0 \quad \textit{and} \quad \rho_n := \tilde{\rho_n}(-\rho_{n+1}) \quad \textit{for } n < 0,$$

we obtain a sequence $(\rho_n)_{n\le 0}$ which is indeed consistent.

Our goal in this section is to extend the sequence $(\rho_n)_{n\le 0}$ to a conditional convex risk measure $\rho_{-\infty}$ with respect to the tail field, to show that this risk measure is continuous from above, and to identify its dual representation. To this end we will make use of the supermartingale properties implied by the consistency condition (14), as they are stated in Theorem 2 and Remark 6.

Theorem 3. *Let us assume*

$$\sup_{n\le 0} E_P[\alpha_n(P)] < \infty. \tag{16}$$

Then the limit

$$\rho_{-\infty}(X) := \lim_{n\downarrow -\infty} \rho_n(X)$$

exists P-a.s. and in $L^1(P)$ for all $X \in L^\infty$. Moreover, the resulting map

$$\rho_{-\infty} :\ L^\infty \to L^\infty(\Omega, \mathcal{F}_{-\infty}, P)$$

defines a conditional convex risk measure with respect to the tail-field $\mathcal{F}_{-\infty}$, and it satisfies the consistency condition

$$\rho_{-\infty} = \rho_{-\infty}(-\rho_n) \tag{17}$$

for all $n \le 0$.

Proof. Fix $X \in L^\infty$. Due to our assumption (16), Theorem 2 together with Remark 6 shows that $(\alpha_n(P))_{n\leq 0}$ is a backwards supermartingale under P which is bounded in $L^1(\Omega, \mathcal{F}, P)$. In view of part (4) of Theorem 2, the same is true for the process

$$V_n(P,X) := \rho_n(X) + \alpha_n(P), \qquad n \leq 0,$$

since it is bounded from below by $-\|X\|_\infty$ and satisfies

$$\sup_{n\leq 0} E_P\left[V_n(P,X)\right] \leq \|X\|_\infty + \sup_{n\leq 0} E_P\left[\alpha_n(P)\right] < \infty.$$

Applying supermartingale convergence backwards under P, we obtain the existence of finite limits

$$V_{-\infty}(P,X) := \lim_{n\downarrow -\infty} V_n(P,X) \tag{18}$$

and

$$\alpha_{-\infty}(P) := \lim_{n\downarrow -\infty} \alpha_n(P)$$

both P-a.s. and in $L^1(P)$; cf. [23]. This yields the existence of the limit

$$\rho_{-\infty}(X) := \lim_{n\downarrow -\infty} \rho_n(X) = V_{-\infty}(P,X) - \alpha_{-\infty}(P) \tag{19}$$

both P-a.s. and in $L^1(P)$. Moreover, we have $|\rho_{-\infty}(X)| \leq \|X\|_\infty$, and it is easy to check that the resulting map

$$\rho_{-\infty}: \ L^\infty \to L^\infty(\Omega, \mathcal{F}_{-\infty}, P)$$

has the properties of a conditional convex risk measure with respect to the tail field $\mathcal{F}_{-\infty}$, as stated in Definition 1. To prove the consistency property (17) of $\rho_{-\infty}$, note that property (14) of the sequence (ρ_n) implies

$$\rho_{-\infty}(-\rho_n(X)) = \lim_{m\downarrow -\infty} \rho_m(-\rho_n(X)) = \lim_{m\downarrow -\infty} \rho_m(X) = \rho_{-\infty}(X)$$

for any $X \in L^\infty$ and $n \leq 0$. □

In the preceding proof, we can replace the reference measure P by any measure Q belonging to the set

$$\mathcal{Q}_P := \left\{ Q \in \mathcal{M}_1(P) \;\middle|\; Q = P \text{ on } \mathcal{F}_{-\infty}, \ \sup_{n\leq 0} E_Q[\alpha_n(Q)] < \infty \right\}.$$

This yields the following result.

Corollary 2. *For any $Q \in \mathcal{Q}_P$, the limit*

$$\alpha_{-\infty}(Q) := \lim_{n\downarrow -\infty} \alpha_n(Q) \tag{20}$$

exists Q-a.s and in $L^1(Q)$, and we have

$$E_Q\left[\alpha_{-\infty}(Q)\right] = \lim_{n\downarrow -\infty} E_Q[\alpha_n(Q)] < \infty.$$

Let us denote by

$$\alpha_{n,n+1}(Q) := \operatorname*{ess\,sup}_{X \in \mathcal{A}_n \cap L^\infty_{n+1}} E_Q[-X|\mathcal{F}_n]$$

the one-step penalty function of $Q \in \mathcal{M}_1(P)$ for $n \leq 0$; we put $L^\infty_1 := L^\infty$ so that $\alpha_{0,1}(Q) = \alpha_0(Q)$.

Lemma 1. *For any $Q \in \mathcal{Q}_P$ the limit $\alpha_{-\infty}(Q)$ in* (20) *is given by*

$$\alpha_{-\infty}(Q) = E_Q\left[\sum_{l=-\infty}^{0} \alpha_{l,l+1}(Q)|\mathcal{F}_{-\infty}\right], \tag{21}$$

and we have

$$\alpha_{-\infty}(Q) = \lim_{n\downarrow -\infty} E_Q[\alpha_n(Q)|\mathcal{F}_\infty] \tag{22}$$

Q-a.s. and in $L^1(Q)$.

Proof. Iterating condition (3) of Theorem 2 for $l = n, \ldots, -1$, we obtain

$$\alpha_n(Q) = \alpha_{n,n+1}(Q) + E_Q[\alpha_{n+1}(Q)|\mathcal{F}_n] = E_Q\left[\sum_{l=n}^{0} \alpha_{l,l+1}(Q)|\mathcal{F}_n\right] \tag{23}$$

for any $n \leq 0$. Combining monotone convergence with martingale convergence ("Hunt's lemma"), we obtain

$$\alpha_{-\infty}(Q) = \lim_{n\downarrow -\infty} \alpha_n(Q) = E_Q\left[\sum_{l=-\infty}^{0} \alpha_{l,l+1}(Q)|\mathcal{F}_{-\infty}\right]$$

Q-a.s. and in $L^1(Q)$. Moreover, (23) implies

$$E_Q[\alpha_n(Q)|\mathcal{F}_{-\infty}] = E_Q\left[\sum_{l=n}^{0} \alpha_{l,l+1}(Q)\big|\mathcal{F}_{-\infty}\right],$$

and so equation (22) follows by monotone convergence. □

Out next goal is to show that the conditional risk measure $\rho_{-\infty}$ has the Fatou property, and that the minimal penalty function in its dual representation is given by the limits $\alpha_{-\infty}(Q)$ for $Q \in \mathcal{Q}_P$. To this end we consider the functional $\rho_P : L^\infty \to \mathbb{R}$ defined by

$$\rho_P(X) := E_P[\rho_{-\infty}(X)]. \tag{24}$$

Lemma 2. *ρ_P is a convex risk measure, and it has the Fatou property.*

Proof. It is easy to see that ρ_P has the properties of a convex risk measure. It remains to prove the Fatou property. Take $X \in L^\infty$ and a uniformly bounded sequence $(X_k)_{k\in\mathbb{N}}$ such that $X_k \to X$ P-a.s.. For any $n \le 0$, the Fatou property of ρ_n implies that the functional $V_n(P,\cdot)$ has the Fatou property as well. Thus we obtain

$$E_P\left[V_n(P,X)\right] \le \liminf_k E_P\left[V_n(P,X_k)\right] \le \liminf_k E_P\left[V_{-\infty}(P,X_k)\right]$$

for all n, where the last inequality follows from the supermartingale property of $(V_n(P,X_k))_{n\le 0}$. Using the supermartingale convergence in (18), this implies

$$E_P\left[V_{-\infty}(P,X)\right] \le \liminf_k E_P\left[V_{-\infty}(P,X_k)\right].$$

Subtracting $E_P\left[\alpha_{-\infty}(P)\right]$ from both sides and recalling (19), we obtain the Fatou property for ρ_P:

$$E_P\left[\rho_{-\infty}(X)\right] \le \liminf_k E_P\left[\rho_{-\infty}(X_k)\right].$$

□

Theorem 4. *Under our assumptions* (16) *and* (14)*, the conditional convex risk measure* $\rho_{-\infty}$ *has the Fatou property, and it admits the representation*

$$\rho_{-\infty}(X) = \operatorname*{ess\,sup}_{Q\in\mathcal{Q}_P}\left(E_Q\left[-X|\mathcal{F}_{-\infty}\right] - \alpha_{-\infty}(Q)\right), \quad X \in L^\infty$$

in terms of the limiting penalty function $\alpha_{-\infty}$ *in* (20) *and* (21)*. Moreover,* $\alpha_{-\infty}$ *coincides with the minimal penalty function of* $\rho_{-\infty}$*, i.e.,*

$$\alpha_{-\infty}(Q) = \operatorname*{ess\,sup}_{X\in\mathcal{A}_{-\infty}} E_Q\left[-X|\mathcal{F}_{-\infty}\right] \quad P\text{-a.s.} \tag{25}$$

for any $Q \in \mathcal{Q}_P$*, where*

$$\mathcal{A}_{-\infty} := \left\{ X \in L^\infty \;\middle|\; \rho_{-\infty}(X) \le 0 \right\}$$

denotes the acceptance set of $\rho_{-\infty}$*.*

Proof. To prove the Fatou property, we show that $\rho_{-\infty}$ is continuous from above. Take $X \in L^\infty$ and a decreasing sequence (X_k) in L^∞ with $X_k \searrow X$ P-a.s.. Monotonicity of $\rho_{-\infty}$ yields

$$\rho_{-\infty}(X) \ge \lim_k \rho_{-\infty}(X_k) \quad P\text{-a.s.}. \tag{26}$$

On the other hand, the unconditional convex risk measure ρ_P in (24) is continuous from above by Lemma 2 and Remark 4. This implies

$$\begin{aligned} E_P\left[\rho_{-\infty}(X)\right] &= \rho_P(X) = \lim_k \rho_P(X_k) = \lim_k E_P\left[\rho_{-\infty}(X_k)\right] \\ &= E_P\left[\lim_k \rho_{-\infty}(X_k)\right], \end{aligned}$$

using monotone convergence in the last step. Combined with (26) this yields

$$\rho_{-\infty}(X) = \lim_k \rho_{-\infty}(X_k) \quad P\text{-a.s.},$$

and hence the Fatou property of $\rho_{-\infty}$. By Corollary 1 it follows that $\rho_{-\infty}$ has the robust representation

$$\rho_{-\infty}(X) = \operatorname*{ess\,sup}_{Q \in \mathcal{Q}_P} \left(E_Q \left[-X | \mathcal{F}_{-\infty} \right] - \tilde{\alpha}_{-\infty}(Q) \right), \quad X \in L^\infty,$$

where we denote by $\tilde{\alpha}_{-\infty}(Q)$ the right-hand side of (25).

For a given $Q \in \mathcal{Q}_P$, we now show that $\tilde{\alpha}_{-\infty}(Q) = \alpha_{-\infty}(Q)$ P-a.s.. Note first that, due to (17) and Theorem 2,

$$\begin{aligned} \tilde{\alpha}_{-\infty}(Q) &= \tilde{\alpha}_{-\infty,n}(Q) + E_Q \left[\alpha_n(Q) | \mathcal{F}_{-\infty} \right] \\ &\geq E_Q \left[\alpha_n(Q) | \mathcal{F}_{-\infty} \right] \qquad Q\text{-a.s.} \end{aligned}$$

for any $n \leq 0$, where $\tilde{\alpha}_{-\infty,n}(Q) \geq 0$ denotes the minimal penalty function of $\rho_{-\infty}$ restricted to $\mathcal{F}_n$. This implies

$$\tilde{\alpha}_{-\infty}(Q) \geq \alpha_{-\infty}(Q) \quad P\text{-a.s.}, \tag{27}$$

using equation (22) in Lemma 1 and the equality of Q and P on $\mathcal{F}_{-\infty}$. To obtain the converse inequality, take any $X \in L^\infty$. We have

$$\rho_n(X) \geq E_Q \left[-X | \mathcal{F}_n \right] - \alpha_n(Q) \quad Q\text{-a.s.}$$

for any $n \leq 0$ by Remark 2, and hence

$$\begin{aligned} \rho_{-\infty}(X) &= \lim_n \rho_n(X) \geq \lim_n \left(E_Q \left[-X | \mathcal{F}_n \right] - \alpha_n(Q) \right) \\ &= E_Q \left[-X | \mathcal{F}_{-\infty} \right] - \alpha_{-\infty}(Q) \quad Q\text{-a.s..} \end{aligned}$$

Since $Q = P$ on $\mathcal{F}_{-\infty}$, we obtain

$$\rho_{-\infty}(X) \geq E_Q \left[-X | \mathcal{F}_{-\infty} \right] - \alpha_{-\infty}(Q) \quad P\text{-a.s..} \tag{28}$$

This holds for all $Q \in \mathcal{Q}_P$, and so we get

$$\begin{aligned} \rho_{-\infty}(X) &= \operatorname*{ess\,sup}_{Q \in \mathcal{Q}_P} \left(E_Q \left[-X | \mathcal{F}_{-\infty} \right] - \tilde{\alpha}(Q)_{-\infty} \right) \\ &\leq \operatorname*{ess\,sup}_{Q \in \mathcal{Q}_P} \left(E_Q \left[-X | \mathcal{F}_{-\infty} \right] - \alpha_{-\infty}(Q) \right) \\ &\leq \rho_{-\infty}(X), \end{aligned}$$

where we have used (27) for the first and (28) for the second inequality. The resulting equality shows that $\rho_{-\infty}$ has a robust representation with penalty function $\alpha_{-\infty}$. Since $\tilde{\alpha}_{-\infty}$ is the minimal penalty function, we obtain $\tilde{\alpha}_{-\infty}(Q) \leq \alpha_{-\infty}(Q)$ P-a.s. for any $Q \in \mathcal{Q}_P$. Combined with (27), this yields equality (25).
□

Thus we have shown backwards convergence of ρ_n as $n \to -\infty$ to a nice conditional risk measure $\rho_{-\infty}$ with respect to the tail field. This can be seen as a backward analogue to the properties of asymptotic safety and asymptotic precision in the forward direction for a consistent sequence $(\rho_n)_{n\geq 0}$ along a filtration $(\mathcal{F}_n)_{n\geq 0}$; see the discussion in [17, Section 5]. As shown in [17, Theorem 5.4], asymptotic safety can be characterized in terms of various asymptotic properties of acceptance sets and of penalty functions as n tends to ∞. The following corollary states backward analogues to those properties as n tends to $-\infty$.

For $n \leq 0$, we denote by

$$\mathcal{A}_{-\infty,n} := \mathcal{A}_{-\infty} \cap L^\infty_n$$

the acceptance set of $\rho_{-\infty}$ restricted to $\mathcal{F}_n$, and by

$$\alpha_{-\infty,n}(Q) := \operatorname*{ess\,sup}_{X\in\mathcal{A}_{-\infty,n}} E_Q\left[-X|\mathcal{F}_n\right],$$

the corresponding minimal penalty function for $Q \in \mathcal{Q}_P$.

Corollary 3. *1.* $\bigcap_n \mathcal{A}_{-\infty,n} = L^\infty_+(\mathcal{F}_{-\infty})$.

2. $\lim_{n\downarrow-\infty} \alpha_{-\infty,n}(Q) = 0$ *Q-a.s. for all $Q \in \mathcal{Q}_P$.*

(3) A position $X \in L^\infty$ belongs to $\mathcal{A}_{-\infty}$ if and only if there exists a uniformly bounded sequence $X_n \in \mathcal{A}_n$, $n \leq 0$, such that $\exists \lim_{n\downarrow-\infty} X_n \leq X$.

Proof. Since each ρ_n and hence $\rho_{-\infty}$ is normalized, we obtain $L^\infty_+(\mathcal{F}_{-\infty}) \subseteq \mathcal{A}_{-\infty,n}$ for any $n \leq 0$, and this shows the inclusion "$\supseteq$" in (1). Conversely, if $X \in \mathcal{A}_{-\infty}$ is $\mathcal{F}_n$-measurable for all $n \leq 0$, then X is $\mathcal{F}_{-\infty}$-measurable, and conditional cash invariance of $\rho_{-\infty}$ yields $-X = \rho_{-\infty}(X) \leq 0$ P-a.s.. This proves (1).

By Theorem 2 combined with (25), the consistency relation

$$\rho_{-\infty}(-\rho_n) = \rho_{-\infty}$$

in Theorem 3 implies

$$\alpha_{-\infty}(Q) = \alpha_{-\infty,n}(Q) + E_Q\left[\alpha_n(Q)|\mathcal{F}_{-\infty}\right]$$

for any $Q \in \mathcal{Q}_P$, and so the convergence in (2) follows from the second equality in Lemma 1.

In order to prove (3), take $X \in \mathcal{A}_{-\infty}$. Note that $X_n := X + \rho_n(X) \in \mathcal{A}_n$ for all $n \leq 0$, that the sequence (X_n) is uniformly bounded by $2\|X\|_\infty$, and that $\lim_n X_n = X + \rho_{-\infty}(X) \leq X$. Conversely, let $X \in L^\infty$ satisfy the condition $X \geq \lim_n X_n$ for some uniformly bounded sequence (X_n) such that $X_n \in \mathcal{A}_n$ for each $n \leq 0$. Since $\rho_n(X_n) \leq 0$, monotonicity and the Fatou property of $\rho_{-\infty}$ together with the consistency condition $\rho_{-\infty}(-\rho_n) = \rho_{-\infty}$ yield

$$\begin{aligned}\rho_{-\infty}(X) &\leq \rho_{-\infty}(\lim_n X_n) \leq \liminf_n \rho_{-\infty}(X_n)\\ &= \liminf_n \rho_{-\infty}(-\rho_n(X_n)) \leq 0,\end{aligned}$$

and so X belongs to $\mathcal{A}_{-\infty}$. □

For the rest of this section we focus on the special case where each ρ_n is *coherent*. Let us denote by $\mathcal{M}_1^e(P)$ the class of all probability measures Q on $(\Omega, \mathcal{F})$ which are equivalent to P.

Definition 4. *A class $\mathcal{Q} \subseteq \mathcal{M}_1^e(P)$ of probability measures on $(\Omega, \mathcal{F})$ is called* stable *with respect to the filtration $(\mathcal{F}_n)_{n\le 0}$ if, for any $Q_1, Q_2 \in \mathcal{Q}$ and any $n \le 0$, the probability measure Q defined by*

$$E_Q[X] := E_{Q_1}[E_{Q_2}[X|\mathcal{F}_n]]$$

belongs again to $\mathcal{Q}$.

Corollary 4. *Under assumptions* (16) *and* (14)*, the conditional risk measure $\rho_{-\infty}$ defined in Theorem 3 is coherent if and only if each ρ_n, $n \le 0$, is coherent. In this case both ρ_n and $\rho_{-\infty}$ have robust representations in terms of the set*

$$\mathcal{Q}_P^e := \mathcal{Q}_P \cap \mathcal{M}_1^e(P) = \left\{ Q \approx P \,\middle|\, \alpha_{-\infty}(Q) = 0 \right\},$$

i.e.,

$$\rho_n(X) = \operatorname*{ess\,sup}_{Q \in \mathcal{Q}_P^e} E_Q[-X|\mathcal{F}_n], \qquad X \in L^\infty$$

for $n \le 0$ and $n = -\infty$. Moreover, the set $\mathcal{Q}_P^e$ is stable, and the process $(\rho_n(X))_{n\le 0}$ is a backward Q-supermartingale for any $Q \in \mathcal{Q}_P^e$.

Proof. It is straightforward to see that the limiting risk measure $\rho_{-\infty}$ is coherent, if each ρ_n is coherent. The converse as well as all other statements of the corollary follow from Theorem 4 and [17, Corollary 4.12], due to the consistency condition (17). □

Remark 8. *Under an additional condition of* comonotonicity*, a conditional version of the arguments in [19, Corollary 4.95] shows that the coherent risk measures ρ_n can be interpreted as conditional Choquet integrals*

$$\rho_n(X) = \int (-X) dC_n,$$

that is, as Choquet integrals with respect to the conditional submodular capacity C_n defined by

$$C_n(A) := \rho_n(-I_A)$$

for any $A \in \mathcal{F}$; see also [7] and the references therein. If these conditional Choquet integrals are consistent in the sense of (14)*, then Theorem 4 shows that they converge along our decreasing σ-fields, that is,*

$$\lim_n \int X dC_n = \int X dC_{-\infty} \qquad P\text{-a.s.}$$

for any $X \in L^\infty$.

4 The structure of global risk measures consistent with $(\rho_n)_{n\le 0}$

Let us denote by $\mathcal{R}$ the class of all convex risk measures ρ on L^∞ which are consistent with the sequence $(\rho_n)_{n\le 0}$, that is, ρ satisfies the condition

$$\rho = \rho(-\rho_n), \qquad n \le 0. \tag{29}$$

Note that $\mathcal{R} \neq \emptyset$, since the risk measure ρ_P defined in (24) belongs to $\mathcal{R}$.

From now on we focus on risk measures $\rho \in \mathcal{R}$ which have the Lebesgue property. We denote be $\mathcal{R}_L$ the class of all those risk measures, and by $\mathcal{R}_{L,S}$ the subclass of all $\rho \in \mathcal{R}_L$ which are strongly sensitive in the sense of Definition 3. In this section, our aim is to clarify the structure of the sets $\mathcal{R}_L$ and $\mathcal{R}_{L,S}$.

Lemma 3. *For any $\rho \in \mathcal{R}_L$ the consistency condition* (29) *extends to the tail field $\mathcal{F}_{-\infty}$, that is,*

$$\rho(-\rho_{-\infty}) = \rho.$$

Proof. For any $X \in L^\infty$ the sequence $(\rho_n(X))_{n\le 0}$ is uniformly bounded by $\|X\|_\infty$ and P-a.s. convergent to $\rho_{-\infty}(X)$. Combining (29) with the Lebesgue property of ρ, we obtain

$$\rho(-\rho_{-\infty}(X)) = \lim_n \rho(-\rho_n(X)) = \rho(X). \tag{30}$$

□

Proposition 2. *We have $\mathcal{R}_{L,S} \neq \emptyset$ if and only if the conditional risk measure $\rho_{-\infty}$ has the Lebesgue property and is strongly sensitive in the sense of Definition 3.*

Proof. Suppose that $\mathcal{R}_{L,S} \neq \emptyset$. For any $\rho \in \mathcal{R}_{L,S}$, we have $\rho(-\rho_{-\infty}) = \rho$ due to Lemma 3. Applying Proposition 1 to ρ and to $\mathcal{F}_1 := \mathcal{F}_{-\infty}$ we see that $\rho_{-\infty}$ has the Lebesgue property and is strongly sensitive.

Conversely, the Lebesgue property of $\rho_{-\infty}$ implies that the risk measure ρ_P defined in (24) belongs to $\mathcal{R}_L$. If, moreover, $\rho_{-\infty}$ is strongly sensitive in the sense of Definition 3, then $\rho := \rho_P$ is strongly sensitive as well. □

Our description of the risk measures in $\mathcal{R}_L$ and $\mathcal{R}_{L,S}$ will involve the conditional risk measure $\rho_{-\infty}$ with respect to the tail filed $\mathcal{F}_{-\infty}$ and an unconditional risk measure on the tail field. More precisely, let us denote by $\hat{\mathcal{R}}$ the class of convex risk measures $\hat\rho$ on $\hat L^\infty := L^\infty(\Omega, \mathcal{F}_{-\infty}, P)$ which have the Lebesgue property on $\hat L^\infty$, and by $\hat{\mathcal{R}}_{L,S}$ the subclass of all $\hat\rho \in \hat{\mathcal{R}}_L$ which are strongly sensitive on $\hat L^\infty$.

Theorem 5. *Suppose that $\rho_{-\infty}$ has the Lebesgue property. Then the class R_L has the following structure:*

$$\mathcal{R}_L = \left\{ \hat\rho(-\rho_{-\infty}) \;\middle|\; \hat\rho \in \hat{\mathcal{R}}_L \right\}. \tag{31}$$

Proof. Take any $\rho \in \mathcal{R}_L$, and denote by $\hat{\rho}$ the restriction of ρ to $\hat{L}^\infty$. Clearly, $\hat{\rho}$ belongs to $\hat{\mathcal{R}}_L$, and Lemma 3 implies

$$\rho = \rho(-\rho_{-\infty}) = \hat{\rho}(-\rho_{-\infty}).$$

This shows the inclusion "$\subseteq$" in (31).

Conversely, take any $\hat{\rho} \in \hat{\mathcal{R}}_L$. Then $\rho := \hat{\rho}(-\rho_{-\infty})$ defines a convex risk measure on L^∞. We have $\rho \in \mathcal{R}$ since

$$\rho(-\rho_n(X)) = \hat{\rho}(-\rho_{-\infty}(-\rho_n(X))) = \hat{\rho}(-\rho_{-\infty}(X)) = \rho(X)$$

due to (17). Moreover, ρ has the Lebesgue property on L^∞. Indeed, for any uniformly bounded sequence (X_k) in L^∞ such that $X_k \to X$ P-a.s., the Lebesgue property of $\rho_{-\infty}$ implies

$$\rho_{-\infty}(X) = \lim_k \rho_{-\infty}(X_k) \quad P\text{-a.s.},$$

hence

$$\rho(X) = \hat{\rho}(-\rho_{-\infty}(X)) = \lim_k \hat{\rho}(-\rho_{-\infty}(X_k)) = \lim_k \rho(X_k)$$

due to the Lebesgue property of $\hat{\rho}$. □

Corollary 5. *If $\mathcal{R}_{L,S} \neq \emptyset$, then*

$$\mathcal{R}_{L,S} = \left\{ \hat{\rho}(-\rho_{-\infty}) \mid \hat{\rho} \in \hat{\mathcal{R}}_{L,S} \right\}.$$

Proof. By Proposition 2, the existence of some $\rho \in \mathcal{R}_{L,S}$ implies that $\rho_{-\infty}$ has the Lebesgue property and is strongly sensitive. For any $\rho \in \mathcal{R}_{L,S}$, the restriction $\hat{\rho}$ of ρ to $\hat{L}^\infty$ clearly belongs to $\hat{\mathcal{R}}_{L,S}$, and we have $\rho = \hat{\rho}(-\rho_{-\infty})$ due to (30).

Conversely, take $\hat{\rho} \in \hat{\mathcal{R}}_{L,S}$. Then Theorem 5 shows that $\rho := \hat{\rho}(-\rho_{-\infty})$ belongs to $\mathcal{R}_L$. Moreover, ρ is strongly sensitive. Indeed, for X and Y in L^∞ with $X \leq Y$ P-a.s.and $P[X < Y] > 0$, we obtain $\rho_{-\infty}(X) \geq \rho_{-\infty}(Y)$ P-a.s.and $P[\rho_{-\infty}(X) > \rho_{-\infty}(Y)] > 0$ due to Proposition 2, and so the strong sensitivity of $\hat{\rho}$ implies

$$\rho(X) = \hat{\rho}(-\rho_{-\infty}(X)) > \hat{\rho}(-\rho_{-\infty}(Y)) = \rho(Y).$$

□

Remark 9. *Throughout this paper, we have worked with a fixed probability measure P. In the spatial setting of [16], the single measure P will be replaced by a whole class $\mathcal{P}$ of probability measures, namely the class of Gibbs measures with given local conditional probabilities. This will require a refined analysis, where conditional risk measures with respect to P are replaced by risk kernels, and where the results of the present paper will be used as building blocks.*

Acknowledgments

Convex risk measures contain, as a special case, the class of Choquet integrals with respect to a submodular Choquet capacity. In this sense they are related to probabilistic potential theory; see, e.g., [21] for a discussion of Choquet capacities in this context. Masatoshi Fukushima and the first author were both working in this area at the time when they first met at the Sixth Berkeley Symposium on Mathematical Statistics and Probability in 1970. It is a great pleasure to dedicate this paper to Masatoshi Fukushima on the occasion of his 80th birthday.

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