Program for the Arbeitsgemeinschaft

“Motives, foliations and the conservativity conjecture”

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Contents

Introduction

The aim of this Arbeitsgemeinschaft is to study the recent proof of the conservativity conjecture for Chow motives in characteristic 0 [11].

Theorem A. ***(Conservativity conjecture for Chow motives)***
Let k be a field of characteristic 0. A morphism of Chow motives over k with rational coefficients is an isomorphism if and only if it induces an isomorphism in cohomology.

By the very definition of Chow motives in terms of smooth projective varieties and correspondences between them, Theorem A is a concrete statement predicting the existence of algebraic cycles with prescribed cohomological properties. For instance, it implies Bloch's conjecture on 0-cycles on surfaces: if X is a smooth projective surface over k with $h^{2,0}(X) = 0$, then the Albanese map on 0-cycles of degree 0 is injective.

Using comparison theorems and elementary arguments, Theorem A can be reduced to the case of algebraic de Rham cohomology. The proof of Theorem A, and consequently this program, consists of roughly four parts.

In the first part (Monday morning), Theorem A is related to mixed motives and motivic homotopy theory in the sense of Morel and Voevodsky. In fact, Theorem A follows from Theorem C below, which asserts the existence of an algebra morphism between certain motivic spectra. In Talk 1.1, we will recall the construction of Voevodsky's tensor triangulated category $\mathbf{DA}^{\text{ét}}(k,k)$ of mixed motives over k with coefficients in k (the relevant coefficients for the de Rham story). As in [11], we adopt the simplified construction without transfers of [35], which is equivalent to the perhaps more familiar category $\mathbf{DM}(k,k)$ defined by Voevodsky. Objects in $\mathbf{DA}^{\text{ét}}(k,k)$ can be represented by T-spectra of complexes of presheaves of k-vector spaces on the category Sm/k of smooth k-schemes (with T being the presheaf $(\mathbb{G}_{\mathrm{m}},1)\otimes k$), satisfying étale (hyper)descent, $\mathbb{A}^1$-invariance and T-stability.

A particularly interesting T-spectrum, the *de Rham spectrum* $\mathbf{\Omega}$, is obtained by putting together the de Rham complexes of all smooth k-varieties. We follow the typographical convention of [11] and use bold $\mathbf{\Omega}$ for the de Rham spectrum, reserving the plain Ω for the de Rham complex. The de Rham spectrum represents de Rham cohomology, in the sense that the de Rham realisation functor

$$\mathrm{dR}^* : \mathbf{DA}^{\text{ét}}(k,k)^{\mathrm{op}} \to \mathrm{D}(\mathrm{Vect}_k),\ \ M \mapsto \mathrm{RHom}_{\mathbf{DA}^{\text{ét}}(k,k)}(M,\mathbf{\Omega})$$

sends the motive of a variety to its total de Rham cohomology.

By Voevodsky's embedding theorem, the category of Chow motives over k with coefficients in k is equivalent to a full subcategory of $\mathbf{DA}^{\text{ét}}(k,k)$, and in fact of the subcategory $\mathbf{DA}^{\text{ét}}_c(k,k)$ of compact objects. Theorem A admits the following refinement, which also follows from the main result in [11].

Theorem A′. ***(Conservativity conjecture for compact motives)*** *Let k be a field of characteristic 0. The de Rham realisation functor*

$$\mathrm{dR}^* : \mathbf{DA}^{\text{ét}}_c(k,k)^{\mathrm{op}} \to \mathrm{D}^b(\mathrm{Vect}^{\mathrm{fd}}_k)$$

is conservative.

The motive $\mathbf{\Omega}$ has a natural structure of commutative algebra object in $\mathbf{DA}^{\text{ét}}(k,k)$ coming from the wedge product. One can form the cobar construction on $\mathbf{\Omega}$, i.e., the cosimplicial T-spectrum [1]

$$\check{C}^\bullet(\mathbf{\Omega}) = \mathbf{\Omega} \mathrel{\substack{\rightarrow\\ \leftarrow\\ \rightarrow}} \mathbf{\Omega}^{\otimes 2} \dots$$

and its homotopy limit $\mathrm{Holim}_\Delta\, \check{C}^\bullet(\mathbf{\Omega})$, taken with respect to stable $\mathbb{A}^1$, ét-local equivalences, which is also a commutative algebra in $\mathbf{DA}^{\text{ét}}(k,k)$.

Conjecture B. *[6, Conjecture B] The unit morphism*

$$k(0) \longrightarrow \operatorname*{Holim}_{\Delta} \check{C}^\bullet(\mathbf{\Omega})$$

is an isomorphism.

[1] Since the motivic spectrum $\mathbf{\Omega}$ is not projectively cofibrant, this should be done carefully to obtain something homotopy-invariant; we ignore this issue in the introduction.

As we will see in Talk 1.2, this conjecture implies a description of morphism groups in $\mathbf{DA}^{\text{ét}}_c(k,k)$ in terms of morphisms in a category of representations of a derived affine group scheme, the derived de Rham motivic Galois group. The main result of [11] is a partial result towards Conjecture B.

Theorem C. *There exists an algebra morphism*

$$\operatorname*{Holim}_{\Delta} \check{C}^{\bullet}(\mathbf{\Omega}) \longrightarrow \tau_{\geq 0}\mathbf{\Omega}$$

in $\mathbf{DA}^{\text{ét}}(k,k)$ where $\tau_{\geq 0}$ is the truncation functor for the homotopy t-structure on $\mathbf{DA}^{\text{ét}}(k,k)$.

As we will also see in Talk 1.2, Theorems A and A′ follow from Theorem C. The argument draws on classical results on algebraic cycles and motives: the Bloch-Ogus formula for algebraic cycles up to algebraic equivalence, the Voisin-Voevodsky theorem on smash-nilpotence, and (for Theorem A′) the theory of weight structures of Bondarko.

The proof of Theorem C will take up the rest of the week. In the next two parts of the program, the focus is on constructing a certain algebra morphism $\mathbf{\Omega} \to \mathbf{\Omega}^{\text{alg}}$ (see below); only in the last part do we come back to the cobar construction for this $\mathbf{\Omega}$-algebra and "compute" $\operatorname{Holim}_{\Delta} \mathbf{\Omega}^{\text{alg}}$ with methods from motivic homotopy theory and Hodge theory.

In the second part (Monday afternoon, Tuesday, Wednesday morning), we present the language of algebraic foliations and explain how the de Rham spectrum $\mathbf{\Omega}$ exhibits new properties when extended to the foliated context. An algebraic foliation $X/\mathcal{F}$ over k is a k-scheme X together with a distinguished quasi-coherent quotient $\Omega^1_{X/k} \twoheadrightarrow \Omega^1_{X/\mathcal{F}}$ satisfying an integrability property. The category of k-schemes embeds fully faithfully as the subcategory of coarse foliations, that is, those for which $\Omega^1_{X/k} = \Omega^1_{X/\mathcal{F}}$. As in differential geometry, there is a presheaf $\mathcal{O}^{\delta}$ of functions "invariant along the leaves", but the leaves themselves do not in general exists as algebraic subvarieties of X. Algebraic foliations admit a differential calculus which extends the one for schemes. In particular, they admit a full de Rham complex $\Omega^*_{X/\mathcal{F}}$ with $H^0(\Omega^*_{X/\mathcal{F}}) = \mathcal{O}^{\delta}(X)$ and notions of diff-étale and diff-smooth morphisms extending the notions of étale and smooth morphisms of schemes. Informally, the leaves of the source foliation of a diff-étale (resp. diff-smooth) morphism are étale (resp. smooth) over the leaves of the target foliation.

There are two constructions of diff-étale morphisms which play a key role in the proof. First, given a smooth scheme X (or, more generally, a diff-smooth foliation), local coordinates $x_0, \ldots, x_n$ on X, and a regular function $f \in \mathcal{O}(X)$, the X-scheme $\mathbb{A}^{\mathbb{N}^n}_X = X[y^{\mathbf{r}},\ \mathbf{r} = (r_1, \ldots, r_n) \in \mathbb{N}^n]$, which is an infinite dimensional affine space over X, has a natural structure of a foliation diff-étale over X such that

$$d(y^{\mathbf{r}}) = \partial_1^{r_1} \cdots \partial_n^{r_n}(f) \cdot dt_0 + \sum_{i=1}^{n} y^{\mathbf{r}+e_i} \cdot dt_i$$

where $e_1, \ldots, e_n$ are the canonical generators of the monoid $\mathbb{N}^n$. Over this foliation, the function f becomes the derivative of the function $y^{\mathbf{0}}$ with respect to the variable x_0. This method of extracting primitives will be used, as in classical differential geometry, to prove a form of the Poincaré lemma.

Second, given an affine scheme X (or even an affine foliation) and a constructible closed subset Z of X, there is a natural foliation on the completion[2] $\widehat{X}_Z$ (considered as a scheme and not a formal scheme) such that the morphism $\widehat{X}_Z \to X$ becomes diff-étale (Talk 1.4). The foliation $\widehat{X}_Z$ will play the role of a tubular neighbourhood of Z in X. To encapsulate these constructions, we introduce some Grothendieck topologies on categories of foliations, generated by certain classes of diff-étale morphisms (Talk 2.1). One of them is the ψ-ft topology, which for the purpose of this introduction can be reasonably conceived as the topology generating by the usual étale topology, by projections $\mathbb{A}^{\mathbb{N}^n}_X \to X$ extracting a primitive of a regular function f with respect to some local coordinates, and by pairs $(X - Z, \widehat{X}_Z)$ coming from completions of affine foliations. Then we have the ψ-ét (resp. ψ-Nis) topology, generated by the étale (resp. Nisnevich) topology and pairs coming from completions.

The de Rham spectrum $\mathbf{\Omega}$ extends to a foliated de Rham spectrum $\mathbf{\Omega}_{\mathfrak{f}}$ on the category SmFol$/k$ of smooth k-foliations. The T-spectrum $\mathbf{\Omega}_{\mathfrak{f}}$ is a central object in the proof. There are geometric

[2] In fact, as the introduction of infinite dimensional affine spaces above makes clear, we need to consider non-noetherian schemes, and for these a variant called weak completion behaves better.

constructions with foliations which cannot be carried out in the world of schemes and which provide interesting new models of $\mathbf{\Omega}_{\mathfrak{f}}$. We provide an informal discussion here and refer to the rest of the program and to [11] for details.

First, we have a foliated version of the Poincaré lemma: the inclusion $\mathcal{O}^{\delta} \to \Omega_{\mathfrak{f}}$ of discrete functions into the de Rham complex is a ψ-ft-local equivalence of complexes of presheaves on SmFol/k. Because primitives of functions exist ψ-ft locally, the classical proof of the holomorphic Poincaré lemma can be transposed directely to the foliated setting (Talk 1.4).

Second, using foliations on $\mathbb{P}^1$-bundles arising as compactifications of exponential integrable connections, one can reconstruct the foliated de Rham spectrum $\mathbf{\Omega}_{\mathfrak{f}}$ (which like its classical counterpart $\mathbf{\Omega}$ is far from being a suspension spectrum) from the suspension spectrum $\Sigma_T^{\infty}\Omega_{\mathfrak{f}}$ by a process called $\overline{\mathbb{E}}$-localisation (Talk 2.3). This process approximates the $(\overline{\mathbb{P}}^{1,\delta}, \mathbb{A}^1, \psi - \mathrm{ft})$-localisation, an operation which is conceptually easier and will be discussed as motivation in Talk 2.2 but is unsuitable for the proof.

Finally, given a k-scheme S, there is a category $\mathbf{FolDA}^{\text{ét}}(S)$ of relative foliated motives in the ψ-ét topology[3] and weak completions can be used to prove the localisation axiom for $\mathbf{FolDA}^{\text{ét}}(-)$, giving access to the six operation formalism of [8] and in particular to Verdier duality (Talk 2.4).

To summarise, one can think of embedding smooth schemes (with the étale topology) into smooth foliations (with the ψ-ft topology) as a form of controlled analytification, which makes certain complex-analytic and formal-geometric arguments possible while preserving much more algebro-geometric information than analytification.

Combining these results on foliations, one can construct an algebra morphism $\mathbf{\Omega}_{\mathfrak{f}} \to \mathbf{\Omega}'_{\mathfrak{f}}$ on (noetherian finite-dimensional) smooth foliations, which is a first step towards the construction of $\mathbf{\Omega}^{\mathrm{alg}}$. Let us say a word about the spectrum $\mathbf{\Omega}'_{\mathfrak{f}}$. Very roughly, one starts with two diagrams of k-varieties: $\mathbb{A}$, whose components are affine spaces $\mathbb{A}^n_k$, and $\mathbb{E}$, which comes with a morphism $\rho : \mathbb{E} \to \mathbb{A}$ whose components are projection maps $\mathbb{P}^a_k \times \mathbb{A}^n_k \to \mathbb{A}^n_k$. Write also $\tilde{\pi} : \mathbb{A} \to \mathbf{Spec}(k)$ for the structural morphism. The indexing categories of $\mathbb{A}$ and $\mathbb{E}$ are quite involved but not relevant for this introduction. Then one has approximately[4]

$$\mathbf{\Omega}'_{\mathfrak{f}} \approx \tilde{\pi}_* \mathrm{R}_{\psi\text{-ft}} \rho_* \mathrm{R}_{\psi\text{-Nis}} \Sigma_T^{\infty} \mathcal{O}^{\delta}$$

where R_{τ} denotes a levelwise τ-local fibrant replacement functor relative to a topology τ on T-spectra.

In the third part (Wednesday afternoon, Thursday), we "compute" the spectrum $\mathbf{\Omega}'_{\mathfrak{f}}$ and its restriction to schemes in order to obtain a new model $bm\Omega^{\mathrm{alg}}$ of the de Rham spectrum $\mathbf{\Omega}$ (or rather an $\mathbf{\Omega}$-algebra).

First we develop techniques to compute the ψ-ft cohomology of schemes for the presheaves of the form $R^i_{\psi\text{-Nis}} \rho_* (T^{\otimes n} \otimes \mathcal{O}^{\delta})$ appearing in $\mathbf{\Omega}'_{\mathfrak{f}}$. The ψ-ft cohomology is described in terms of ψ-Nis cohomology, using the theory of *malleable* presheaves, which allows to restrict to the much smaller site of foliations obtained from schemes by iterated completions (Talks 3.2 and 3.3). Then the ψ-Nis topology is replaced by the ψ-h topology, a foliated version of the h-topology of Voevodsky, and the ψ-h-cohomology for the presheaves we are interested in is shown to be closely related to their h-cohomology, provided we compute it on flags of subvarieties (Talks 4.1 and 4.2). We refer to the description of the talks for more details. All this leads to an algebra morphism $\mathbf{\Omega} \to \mathbf{\Omega}''$ on the category of finite type k-schemes. Let us give a flavour of what $\mathbf{\Omega}''$ looks like.

Given a complex of presheaves F over Sch/S with S a scheme or diagram of schemes, there is a formal Godement resolution $\Xi(F)$ which is obtained by putting together sections of F on iterated completions $\widehat{X}_{\underline{D}}$ of $X \in \mathsf{Sch}/S$ along a flag $\underline{D}$ of subvarieties in X, and such that $F \to \mathrm{Tot}\,\Xi(F)$ is a ψ-Nis local equivalence. On $\mathsf{Sch}/\mathbb{E}$ with $\mathbb{E}$ the diagram of schemes mentioned above, there is a complex of presheaves $\mathcal{Y}$ which is the de Rham complex of a natural logarithmic integrable connection. Then one has approximately[5]

$$\mathbf{\Omega}'' \approx \tilde{\pi}_* p_* \mathrm{R}_{\mathrm{h}} \Sigma_T^{\infty}\, \mathrm{Tot}\, \Xi(\mathcal{Y})$$

[3]The category $\mathbf{FolDA}^{\text{ét}}(S)$ is the homotopy category of a stable $\mathbb{A}^1, \psi$-ét-local model structure on T-spectra of complexes of presheaves over SmFol/S.

[4]The spectrum $\mathbf{\Omega}'_{\mathfrak{f}}$ we have in mind is actually $\underline{\mathrm{Sg}}^{\mathbb{A}^1} \Lambda^{\infty}\, \mathrm{Tot}^{\mathbb{N},\overline{\Theta}}\, \pi_* \mathrm{R}_{\psi\text{-ft}} \rho_* \mathrm{R}_{\psi\text{-Nis}} \Sigma_T^{\infty} \widetilde{\mathcal{O}}^{\delta}$ as in [11, p. 3.9.3].

[5]The spectrum $\mathbf{\Omega}''$ we have in mind is actually $\underline{\mathrm{Sg}}^{\mathbb{A}^1} \Lambda^{\infty}\, \mathrm{Tot}^{\mathbb{N},\overline{\Theta}}\, \pi_* p_* \mathrm{R}_{\mathrm{hh}^{\star}}\, \mathrm{Tot}\, \Xi(\overline{\Sigma}_T^{\infty} \mathcal{Y})$ as in [11, p. 5.10.18]

where h denotes the h-topology of Voevodsky. At this point, the definition of $\mathbf{\Omega}''$ does not involve foliations and the rest of the argument takes place in complex algebraic geometry and Hodge theory.

The model $\mathbf{\Omega}''$ is not quite suitable for the rest of the proof. In particular, it involves iterated completions which are not schemes of finite type over k, and hence outside of the range of classical Hodge theory. This is remedied in the following way. Because we work locally for the h-topology and we have resolution of singularities, it turns out that there is an alternative, "algebraic" model $\mathbf{\Omega}^{\mathrm{alg}}$ of $\mathbf{\Omega}''$ built using only $\mathbb{E}$-schemes which are smooth over k, and flags which are constructible with respect to strict normal crossing divisors in such schemes (Talks 4.3 and 4.4). Given a pair $(X, \underline{D})$ of such a smooth variety and a normal crossing flag, it turns out that the de Rham cohomology of the iterated completion $\widehat{X}_{\underline{D}}$ can be algebraised: there exists a scheme $M^{\circ}_X(\underline{D}) \to X$ of finite type such that its de Rham cohomology (and more generally the de Rham cohomology of logarithmic connections) is "the same" as that of $\widehat{X}_{\underline{D}}$. In the construction of $\mathbf{\Omega}^{\mathrm{alg}}$, everything is of finite type over k, hence in particular finite-dimensional. In preparation for the end of the proof, those dimensions have to be carefully controlled; this leads after some work to a double dimensional filtration of $\mathbf{\Omega}^{\mathrm{alg}}$.

In the fourth part (Friday), we analyse the homotopy limit $\mathrm{Holim}_\Delta \check{C}^\bullet(\mathbf{\Omega}^{\mathrm{alg}})$. The techniques come from motivic homotopy theory (Bloch-Levine coniveau towers, properties of Voevodsky contractions, $\mathbb{A}^1$-connectivity theorems, vanishing properties of motivic cohomology, etc.) and also from Hodge theory. In particular, we need to use a complex embedding $\sigma : k \hookrightarrow \mathbb{C}$ (and a reduction to the case where k admits such an embedding). Note, however, that Hodge theory is used to show that an algebraically defined map somewhere in the proof is an isomorphism and not to construct something transcendental. Consequently, the morphism constructed in the proof of Theorem C only depends on k and not on the choice of σ.

Let us first explain the relevant ingredient from Hodge theory. Given an Artin neighbourhood X (i.e., a particular type of smooth affine $\mathbb{C}$-variety) and an algebra object $\mathcal{A}$ in variations of mixed Hodge structures over X satisfying an assumption on its weights, then we can form the cobar construction $\check{C}^\bullet(\Gamma(X, \mathcal{A}))$. The conclusion is then that something like Conjecture B holds, namely

$$\mathcal{O}(X) \xrightarrow{\sim} \operatorname*{Holim}_{\Delta} \check{C}^\bullet(\Gamma(X, \mathcal{A})).$$

and this convergence of the homotopy limit is functorial and is controlled by the dimension of X, so that one can replace X by a diagram of Artin neighbourhoods of dimensions $\leq N$ for some fixed N. The proof relies on weight arguments and on the existence of Deligne splittings of the underlying C^∞-vector bundle of a variation of mixed Hodge structures.

By the construction of $\mathbf{\Omega}^{\mathrm{alg}}$ in terms of the de Rham cohomology of certain logarithmic connections, there are natural algebras in variations of Hodge structure appearing over the diagram of schemes defining $\mathbf{\Omega}^{\mathrm{alg}}$. To apply the Hodge-theoretic ingredient, one would like to work only with the subdiagram consisting of schemes of dimension $\leq N$. This does not work on the nose but inspires the following. A motivic spectrum admits a Bloch-Levine coniveau tower, which is a combination of Levine's homotopy coniveau towers [33] and Bloch's "moving by blowups" [16]. An analysis of (a suitable variant of) the Bloch-Levine tower $\mathrm{Holim}_\Delta \check{C}^\bullet(\mathbf{\Omega}^{\mathrm{alg}})$ shows that, when computing the sections of this tower on a given smooth k-scheme X, it is possible to work with a subdiagram of schemes of dimension $\leq N$ (depending on X) and apply the Hodge theoretic ingredient to do a key simplification of the homotopy limit.

There is then a cleaning-up phase of the resulting spectrum, consisting in folding back many of the previous steps into simpler and simpler spectra. This actually involves revisiting some foliated constructions. We end up constructing an algebra morphism $\mathrm{Holim}_\Delta \check{C}^\bullet(\mathbf{\Omega}^{\mathrm{alg}}) \to \mathbf{\Omega}$ which factors through an algebra morphism

$$\operatorname*{Holim}_{\Delta} \check{C}^\bullet(\mathbf{\Omega}) \to \mathscr{B}$$

such that the spectrum $\mathscr{B}$[6] is (-1)-connected. By definition of the homotopy t-structure, this produces a morphism

$$\operatorname*{Holim}_{\Delta} \check{C}^\bullet(\mathbf{\Omega}) \to \tau_{\geq 0}\mathbf{\Omega}.$$

This completes the proof of Theorem C and hence of Theorems A and A′.

[6]The spectrum $\mathscr{B}$ is actually $\widetilde{\Pi}((\Sigma^\infty_T \widetilde{\mathscr{R}})_{\psi\text{-ét}})$ as in [11, p. 8.10.29]

References

The main reference is [11]. Note that the numbering corresponds to the version of April 2018, which is accessible on the website of the Arbeitsgemeinschaft, and not the version on Ayoub's website, which is still under revision. Besides [11], there is a lecture course on the topic in the spring 2018 at the University of Zürich: videos are available from the website

`http://user.math.uzh.ch/ayoub/`

The paper [13] is concerned with another aspect of foliations, namely the relationship between the foliated topology and (higher) Differential Galois Theory. By and large, the two papers [11] and [13] are independent, and only a few introductory sections of [13] are necessary to understand the proof of the conservativity conjecture: the details are in the descriptions of talks 1.3, 1.4, 3.1 and 3.2 below.

Prerequisites

The main prerequisite for the proof is the homotopical algebra underlying the definition and study of triangulated categories of mixed motives. Even though the definition of $\mathbf{DA}^{\text{ét}}(k)$ will be briefly recalled in the first talk, one should be confortable with the following notions.

- Model categories, including the stable, monoidal, dg/simplicial theory,
- Bousfield localisation,
- Local model structures on categories of complexes of (pre)sheaves on a site,
- Spectra in model categories,
- Simplicial objects in additive categories, Dold-Kan correspondence,
- Cubical objects in additive categories, cubical Dold-Kan correspondence,
- Pro- and Ind-categories.

A reference which covers a lot of this background material is [8, Chapitre 4]. One can also look at the first few chapters of the course notes [2].

Another important ingredient which will not be recalled is the basic theory of the de Rham complex, algebraic de Rham cohomology, and the definition of algebraic integrable connections on smooth varieties [24]. Familiarity with valuation rings beyond the case of discrete valuation rings will be very useful starting from Talk 4.1. Also, in a few places, excellent rings and schemes, their relationship with completion, and the fact that in characteristic 0 they admit resolution of singularities [38] will be used.

Familiarity with algebraic cycles, adequate equivalence relations and Chow groups is necessary to understand the statement and relevance of Theorem A, and how to deduce it from Theorem C, but is not necessary to understand the proof of Theorem C.

Preparation of the talks

There is a lot of material to cover in each talk. The program should be interpreted as a very optimistic aim, and there will be compromises to be made! However, the structure of the proof is relatively linear and modular, so that it is often possible to summarize the work done in each talk with one statement which is the only necessary take-away; these statements should always be explained clearly.

Please do not hesitate to ask the organisers for clarification on any aspect of this program!

1 Monday: Motives, cycles and foliations

From now on, all numbered references which are not specified are to [11].

1.1 Voevodsky motives and the de Rham spectrum

The category of motives relevant for [11] is the étale $\mathbb{A}^1$-derived category $\mathbf{DA}^{\text{ét}}(k,R)$, considered as a tensor triangulated category, and in fact as the homotopy category of a certain stable monoidal dg-model category of (symmetric) T-spectra of complexes of presheaves of R-modules (for R a commutative ring) on the category Sm/k of smooth k-schemes. In the case $R = k$, there is a particularly interesting object, the de Rham spectrum $\mathbf{\Omega}$, which represents de Rham cohomology.

The aim of this first talk is to recall the construction of $\mathbf{DA}^{\text{ét}}(k,R)$ and some basic techniques which can be used to study it, with a view towards ideas which will be adapted to the foliated setting. Here are topics which should be covered. Good references for this talk include [2, Chapter I] [1] [34].

- Model categories of spectra in a model category, stable model structure, stabilisation functor Λ^∞ [8, §4.3] [28]. Focus on the case of usual (non-symmetric) spectra, as most of the constructions later will be done in this context. If you want to talk about the monoidal aspects, instead of symmetric spectra, you could introduce commutative spectra [3, §4-5], a simplified version of symmetric spectra which works well in the $\mathbb{Q}$-linear setting. This is the part of the construction of $\mathbf{DA}^{\text{ét}}(k,R)$ which is perhaps least familiar to an audience of algebraic geometers, and spectra are used systematically in the rest of the program.

- Define the stable $(\mathbb{A}^1, \text{ét})$-local projective model structure on motivic spectra with coefficients in a ring R [4, §3] and its homotopy category $\mathbf{DA}^{\text{ét}}(k,R)$. We adopt the model $T = (\mathbb{G}_{\text{m}}, 1) \otimes R$ of 3.1.6 for T. Discuss the alternative models of T as in 2.2.1, 3.1.6; having this flexibility will be used in later talks. Define the motive $M(X) \in \mathbf{DA}^{\text{ét}}(k,R)$ of a smooth variety k-variety X.

- Describe explicitly what it means for a motivic spectrum to be fibrant: Ω_T-spectra, $\mathbb{A}^1$-locality, descent for étale hypercovers. In more concrete cohomological terms, the first two parts are [1, Definition 2.19]; the (hyper)descent condition is then saying that the étale hypercohomology groups in this definition are computed on global sections, by Verdier's hypercovering theorem.

- Assume for the rest of the talk that $R = F$ is a field of characteristic 0. Explain why in this case weak equivalences in the model structure on motivic spectra are stable by filtered colimits. Deduce that the triangulated category $\mathbf{DA}^{\text{ét}}(k,F)$ is compactly generated, and that compact objects coincide precisely with constructible ones [4, pp. 3.16-19].

- Define the Suslin-Voevodsky singular complex construction. Explain that, because of the previous fact about filtered colimits, one has an "explicit" description of the stable $(\mathbb{A}^1, \text{ét})$-localisation functor: if $\mathrm{R}_{\text{ét}}$ denotes a levelwise étale fibrant replacement functor on motivic spectra of k-vector spaces, then the functor $\Lambda^\infty \mathrm{Sing}^{\mathbb{A}^1} \mathrm{R}_{\text{ét}}$ is a stable $(\mathbb{A}^1, \text{ét})$-localisation functor. A reference for this is §3.3 (which treats the foliated analogue, but the argument is exactly the same).

- State Poincaré duality and the closely related fact that compact objects in $\mathbf{DA}^{\text{ét}}(k,F)$ coincide precisely with the strongly dualisable ones (see [37] for the parallel case of the motivic homotopy category).

- (Time permitting) Recall without proof that over a field k of characteristic 0 and with coefficients in a $\mathbb{Q}$-algebra F, we have $\mathbf{DA}^{\text{eff,ét}}(k,F) \simeq \mathbf{DM}^{\text{eff}}(k,F)$ and $\mathbf{DA}^{\text{ét}}(k,F) \simeq \mathbf{DM}(k,F)$ [6, Appendix B] [21]. Mention also that, on the other hand, $\mathbf{DA}^{\text{Nis}}(k,F)$ is a different category and that the use of the étale topology is necessary when working without transfers.

- State Voevodsky's theorem that Chow groups with coefficients in F are represented as morphism groups in $\mathbf{DM}(k,F)$, hence in $\mathbf{DA}^{\text{ét}}(k,F)$ by the previous point. Deduce from this and Poincaré duality that the category of Chow motives is equivalent to the pseudo-abelian completion of the full sub-category of $\mathbf{DA}_c^{\text{ét}}(k,k)$ whose objects are Tate twists of motives of smooth projective varieties over k (see [2, §16, 17], [34, Lectures 19 and 20]).

- De Rham spectrum: describe the general construction of spectra via $\tau^{\leq 1}\Omega_{/k}$-modules (adapt 2.2.3-2.2.9 to the scheme case), construct the de Rham spectrum $\mathbf{\Omega}$ as a motivic symmetric (or commutative) spectrum and skech why the wedge product of differential forms endows it with a commutative algebra structure [6, §2.3.1] [2, Chapter 2 §9].

- The motivic spectrum $\mathbf{\Omega}$ is almost fibrant in the following sense. The complex $\Omega_{/k}$, considered in the ét-local model category of complexes of presheaves, represents de Rham cohomology for smooth varieties: this follows from étale descent for coherent sheaves. Using the Künneth formula for de Rham cohomology and the cohomology of $\mathbb{A}^1$ and $\mathbb{G}_{\mathrm{m}}$, this implies that $\mathbf{\Omega}$ is an Ω_T-spectrum and that it is $\mathbb{A}^1$-local. At this point, we know that the levelwise étale replacement of $\mathbf{\Omega}$ is fibrant; using again étale descent for coherent cohomology, we get that the levelwise Zariski replacement of $\mathbf{\Omega}$ is already fibrant. Some of these arguments are covered in [6, §2.3.1]. Conclude that $\mathbf{\Omega}$ represents de Rham cohomology in the sense that the de Rham realisation functor

$$\mathrm{dR}^* : \mathbf{DA}^{\text{ét}}(k,k)^{\mathrm{op}} \to \mathrm{D}(\mathrm{Vect}_k),\ M \mapsto \mathrm{RHom}_{\mathbf{DA}^{\text{ét}}(k,k)}(M,\mathbf{\Omega})$$

  sends the motive of a variety to its total de Rham cohomology.

- Point out that the motivic spectrum $\mathbf{\Omega}$ is not projectively cofibrant, for instance using the criterion in [29, Proposition 1.14]. Similarly, the corresponding symmetric spectrum is not projectively cofibrant. In particular, computing its derived tensor product with some other motivic spectrum is not straightforward.

- Modules for the de Rham spectrum: sketch the proof of [5, Proposition 2.11], and mention the resulting equivalence of categories between modules over $\mathbf{\Omega}$ and $\mathrm{D}(\mathrm{Vect}_k)$ [22, Theorem 2.6.2].

1.2 Motivic conjectures, main theorem and its corollaries for algebraic cycles

The first part of the talk should present the conjectural picture of derived Tannakian duality for $\mathbf{DA}^{\text{ét}}(k,F)$, without complete proofs, as motivation for the statement of Theorem C [6, §2.4] [3]. The papers [6] [3] are written for the Betti realisation, which is the natural thing to do to obtain a motivic Galois group defined over $\mathbb{Q}$, but with different technical details a similar story works for the de Rham realisation. The first two points below should be done carefully as we will use them in the proof.

- Introduce the cosimplicial cobar construction (also sometimes called the Amitsur resolution) for a commutative algebra object in a monoidal model category. A possible reference is [19, §2-3], which is done in the setting of classical stable homotopy theory but whose definition extend. Note that for a cofibrant algebra, the construction is homotopy invariant. Since $\mathbf{\Omega}$ is not a cofibrant motivic spectrum, one has to take a projectively cofibrant replacement first. See the discussion at the beginning of §8.15.

- Recall the definition of the homotopy limit of a tower, i.e. an $\mathbb{N}^{\mathrm{op}}$-indexed object, in a simplicial model category. Similarly, recall the definition of the homotopy limit of a cosimplicial object in a simplicial model category. Explain the relationship between the two for model categories enriched over model categories of complexes (see Remarque 8.7.4). For introductions to the general theory of homotopy limits, one can consult [36, Part I] or [27, Chapter 18]; see also [19, §2.2]. We do not need to know much about these constructions, basically enough to substantiate the arguments at the beginning of §8.7.

- Present briefly the weak Tannakian formalism of [6, §1] and its model category level version in [3, §3]; the relation between the two is explained in [3, Theorem 3.12]. Point out the key hypothesis [6, Hypothese 1.40 (b)] [3, Setting 3.1 (c)].

- We want to apply the formalism to the adjunction $\mathrm{dR}^* : \mathbf{DA}^{\text{ét}}(k,k)^{\text{op}} \to \mathrm{D}(\mathrm{Vect}_k) : \mathrm{dR}_*$, where here dR^* refers to the covariant de Rham realisation; the right adjoint exists by general principles (Neeman representability for compactly generated triangulated categories) but can also be described more explicitely: $\mathrm{dR}_* K = \mathbf{\Omega} \otimes K$, using the enrichement of the model category of motivic spectra over the model category of complexes. Explain that the last result of the previous talk allows one to check the hypotheses of the weak Tannakian formalism in this case. Introduce the de Rham motivic Hopf algebra $\mathcal{H}^{\mathrm{dR}}_{\mathrm{mot}}(k)$ (which is an Hopf algebra object in $\mathrm{D}(\mathrm{Vect}_k)$), its homotopical enhancement $\underline{\mathcal{H}}^{\mathrm{dR}}_{\mathrm{mot}}(k)$ (which is an homotopy Hopf algebra in a model category of complexes) and their relationship [3, Theorem 3.12].

- State that the Grothendieck comparison theorem between Betti and de Rham cohomology and results on the Betti realisation from [6] imply that the complex underlying $\mathcal{H}^{\mathrm{dR}}_{\mathrm{mot}}(k)$ is (-1)-connected. Use this to introduce the de Rham motivic Galois group $G^{\mathrm{dR}}_{\mathrm{mot}}(k)$ which is an affine group scheme over k [6, Corollaire 2.105, Definition 2.106]. Explain, following [3, Introduction], that the de Rham realisation lifts to a triangulated functor
$$\mathbf{DA}^{\text{ét}}(k,k) \longrightarrow \mathrm{D}(\mathrm{Ind}\,\mathrm{Rep}^{\mathrm{fd}}(G^{\mathrm{dR}}_{\mathrm{mot}}(k))).$$

- Present the two conjectures (one of which is our Conjecture B) of [6, §2.4], and state that their combination implies that the functor above restricts to an equivalence of categories
$$\mathbf{DA}^{\text{ét}}_c(k,k) \simeq \mathrm{D}^b(\mathrm{Rep}^{\mathrm{fd}} G^{\mathrm{dR}}_{\mathrm{mot}}(k))$$
which is a precise form of the motivic t-structure conjecture. In particular, explain that assuming all the conjectures, the statement of Conjecture B is simply the statement that the "standard resolution" of the trivial representation of $G^{\mathrm{dR}}_{\mathrm{mot}}(k)$ by tensor powers of the regular representation is a resolution.

- (Time permitting) Another completely different construction of the motivic Galois group attached to a cohomology theory is due to Nori. Recall that by [20], this yields the same group (for the Betti realisation, should also be true for de Rham) as the construction of [6].

The second part of the talk should state Theorem C, derive the conservativity conjecture, and describe some concrete consequences for algebraic cycles.

- Restate Theorems A and A′. Explain that Theorem A′ would follow from the conjectural picture of the first half of the talk and implies Theorem A by Voevodsky's embedding. Observe that Theorem A′ (and hence cases of Theorem A) is known on the triangulated subcategory of $\mathbf{DA}^{\text{ét}}_c(k,k)$ of motives of abelian type (a result due to Wildeshaus [40] [9, Corollary 2.3]).

- Notice that the de Rham realisation is not conservative on the large category $\mathbf{DA}^{\text{ét}}(k,k)$ [9, Lemma 2.4].

- (Time permitting) Explain how to reduce Theorems A and A′ (which make sense for the Betti, ℓ-adic, de Rham realisations) to the case of the de Rham realisation and $k = \mathbb{C}$.

- (Time permitting) Sketch the proof of Bloch's conjecture for surfaces assuming conservativity for Chow motives [9, Proposition 2.15].

- Describe the standard t-structure on motivic spectra (paying special attention to our choice of $T = (\mathbb{G}_{\mathrm{m}}, 1) \otimes k$) and state Morel's theorem that it descends to a t-structure on $\mathbf{DA}^{\text{ét}}(k,k)$ called the homotopy t-structure (see [2, Chapter 1 §11] for the effective case; for spectra, one could adapt the discussion of [23, §5] to the context without transfers, taking note that we are using an homological convention). Explain that the t-structure is compatible with the monoidal structure on symmetric motivic spectra and deduce in particular that $\tau_{\geq 0}\mathbf{\Omega} = (\tau_{\geq -n}\Omega_{/k}[n])_{n\in\mathbb{N}}$ has a natural algebra structure in $\mathbf{DA}^{\text{ét}}(k,k)$.

- State Theorem C (correctly, using a cofibrant replacement of $\mathbf{\Omega}$; see the discussion at the beginning of 8.7). In fact, in order to simplify our job this week and not have to keep track of monoidal structures and systematically work with symmetric spectra, we will focus on the slightly weaker version below, which is enough for the conservativity conjecture, and which is the statement proved in the current version of [11]. For this, one needs the fact that homotopy limits of cosimplicial objects only depend on the underlying semi-cosimplicial object (8.7.4).

Theorem D. *Fix a projectively cofibrant replacement* $\mathbf{\Omega}_{\mathrm{cof}}$ *of* $\mathbf{\Omega}$*. There exists a morphism*

$$\operatorname*{Holim}_{\Delta} \check{C}^{\bullet}(\mathbf{\Omega}_{\mathrm{cof}}) \longrightarrow \tau_{\geq 0}\mathbf{\Omega}$$

where $\tau_{\geq 0}$ *is a truncation functor for the homotopy t-structure on* $\mathbf{DA}^{\text{ét}}(k,k)$*, which is unital, i.e., compatible with the unit maps of the algebra structures.*

- Using the definition of the homotopy t-structure for T-spectra, Morel's $\mathbb{A}^1$-connectivity theorem and the fact that $\mathbf{\Omega}$ is almost fibrant as explained above, prove that, given a smooth variety X over k and $n \geq 0$, we have

$$\mathbf{DA}^{\text{ét}}(k)(M(X), (\tau_{\geq 0}\mathbf{\Omega})(n)[2n]) \simeq H^n(X_{\mathrm{Zar}}, (\tau_{\geq -n}\Omega_{/k})[n]).$$

- Using [17, Corollary 6.2] and the Bloch-Ogus formula [17, Corollary 7.4,7.6], prove that in fact

$$H^n(X_{\mathrm{Zar}}, (\tau_{\geq -n}\Omega_{/k})[n]) \simeq H^n(X_{\mathrm{Zar}}, \mathcal{H}^n_{\mathrm{dR}}) \simeq \mathrm{CH}^n_{\mathrm{alg}}(U)_k.$$

- Recall the Voisin-Voevodsky nilpotence theorem [39, Corollary 3.3,3.4] and why it implies that the functor $\mathbf{Chow}(k,k) \to \mathbf{Chow}_{\mathrm{alg}}(k,k)$ from Chow motives to pure motives up to algebraic equivalence is conservative.

- Using the two results above and [5, Proposition 2.11], prove that Theorem A follows from Theorem D as in the end of [11, Introduction]. Point out where the compactness hypothesis is used and how this does not contradict the failure of conservativity on the larger category $\mathbf{DA}^{\text{ét}}(k,k)$.

- (Time permitting) Present the argument due to Bondarko that Theorem C implies Theorem A$'$. Here is a sketch. The theory of weight structures of Bondarko [18] can be applied both to $\mathbf{DA}^{\text{ét}}_c(k,k)$ (where Bondarko constructed a weight structure with heart $\mathbf{Chow}(k,k)$) and to the category of compact $\tau_{\geq 0}\mathbf{\Omega}$-modules (where Bondarko's method yields a weight structure with heart $\mathbf{Chow}_{\mathrm{alg}}(k,k)$). These weight structures come with conservative weight complex functors $\mathbf{DA}^{\text{ét}}_c(k,k) \to \mathrm{K}(\mathbf{Chow}(k,k))$ and $\tau_{\geq 0}\mathbf{\Omega} - \mathrm{Mod_c} \to \mathrm{K}(\mathbf{Chow}_{\mathrm{alg}}(\mathrm{k,k}))$, where $\mathrm{K}(-)$ denotes homotopy categories of additive categories. The natural functor $\mathrm{K}(\mathbf{Chow}(k,k)) \to \mathrm{K}(\mathbf{Chow}_{\mathrm{alg}}(k,k))$ is also conservative (this also follows from Voisin-Voevodsky). At this point looking hard at a natural commutative diagram is enough deduce Theorem A$'$ from Theorem C.

- State and sketch the proof of Lemma 8.7.6.

1.3 Foliations I: elementary geometry

We develop the basic geometry of foliations, including their de Rham complexes and the notions of diff-étale and diff-smooth morphisms. Besides [11, §1], one can look at [10, Chapter I].

- Present §1.1 up to Proposition 1.18.

- Present §1.2 up to Proposition 1.2.20. This requires presenting some material from [13] on Δ-schemes (and hence also on basic differential algebra: Δ-rings and Δ-modules). Proposition 1.2.21 should be stated but not proven.

1.4 Foliations II: integrable connections and the Poincaré lemma

Integrable connections provide fundamental examples of diff-étale morphisms of foliations and will be used in many parts of the proof.

- Present the end of §1.1, starting from Definition 1.1.19.
- Present [13, §5.10.1-4,5.11.1-2], ignoring the constructions for arbitrary commutative algebraic groups.
- Prove the foliated Poincaré lemma 2.5.18, formulated not in the context of a Grothendieck topology but as the concrete statement about de Rham cohomology classes in the last sentence of the first paragraph of the proof. Point out the analogy with the holomorphic Poincaré lemma.
- Present the definition and properties of logarithmic connections in 2.5.7-10.
- Explain the construction of (compactified) exponential connections (2.1.20).

The rest of the talk is background material which will be used in many places starting from Talk 3.4.

- Recall the basics of the theory of regular singular integrable connections on smooth k-varieties [25, §II], their stability properties (extensions, tensor products, pullbacks) and the comparison theorem (i.e., the full faithfullness part of the Riemann-Hilbert correspondence for local systems) when $k = \mathbb{C}$ [25, §II.6, Théorème 6.2]. As examples, observe that logarithmic connections are automatically regular singular, while exponential connections are in general not regular singular.
- Recall that if $f : X \to S$ is a smooth projective morphism between smooth k-varieties and $(\mathcal{E}, \nabla)$ is a regular singular connection on X, then the Gauss-Manin connections on $R^i f_* \mathcal{E}$ are also regular singular [30].

2 Tuesday: foliated topology, exponential connections, motivic Verdier duality

2.1 Foliations III: weak completions and topologies

The morphism $\mathbf{Spec}(k[[t]]) \to \mathbf{Spec}(k[t])$ is not an étale morphism of schemes, essentially because for a power series f, the relation $df = f'dt$, with f' the formal derivative of f, does not hold in $\Omega^1_{k[[t]]/k}$. In the foliated setting, however, we can impose this relation and define a foliation on $\mathbf{Spec}(k[[t]])$ such that the morphism $\mathbf{Spec}(k[[t]]) \to \mathbf{Spec}(k[t])$ becomes diff-étale. This construction works more generally for constructible closed subsets of affine schemes, and leads to the notion of (weak) completion of an affine foliation. The term “weak” refers to the fact that ordinary completion of rings along an ideal does not have good properties outside of the noetherian setting, and we need to use a variant which does. Weak completions can be thought of as substitutes for tubular neighbourhoods in the foliated setting.

- Present the material of §1.3-1.4. Notice that, given a noetherian finite dimensional affine scheme X and a closed subscheme Z, the weak completion of X along Z is still noetherian finite dimensional.

We now come to the definition of the various Grothendieck topologies on foliations which occur in the proof.

- Present the material of §1.9, i.e., define the ψ-ft topology, a Grothendieck topology on foliations which is tailored to take advantage of the foliated Poincaré lemma and of weak completions, and its coarser variants ψ-ét and ψ-Nis. This requires going back to section 1.5-1.6 and adapting the proofs there; they are written in terms of the foliated topology, a finer topology on foliations which is somewhat more natural than the ψ-foliated one, but which is too fine for the $\overline{\mathbb{E}}$-localisation arguments of the next talk.

- Reinterpret the foliated Poincaré lemma (2.5.18) using the ψ-ft topology and explain the analogous result in complex analytic geometry.
- Explain the results of §1.7, which construct a certain family of points of the ψ-ft topos. This is not a conservative family, but it is enough to do certain computations on excellent foliations in Talk 3.2.

2.2 $\overline{\mathbb{E}}$-localisation I

- Sketch very quickly why the analogue of Conjecture B holds if $\mathbf{\Omega}$ is replaced by $\Sigma^\infty_T \Omega_{/k}$ (as in the Introduction). This will be revisited later on, and should just be thought of as motivation for trying to relate our problem to the suspension spectrum $\Sigma^\infty_T \Omega_{/k}$.

Motivated by the above, we want a model of $\mathbf{\Omega}_{\mathfrak{f}}$ built from the suspension spectrum $\Sigma^\infty_T \Omega_{\mathfrak{f}}$. Consider the discrete foliation $\mathbb{P}^{1,\delta}$ with underlying scheme $\mathbb{P}^1$ and $\Omega^1_{\mathbb{P}^{1,\delta}/k} = 0$. There is a natural notion of $(\mathbb{P}^{1,\delta}, \text{ét})$-local spectrum, and because $H^*(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}) \simeq k[0]$, the spectrum $\mathbf{\Omega}_{\mathfrak{f}}$ is $(\mathbb{P}^{1,\delta}, \text{ét})$-local. There is a finer notion of $(\mathbb{P}^{1,\delta}, \psi\text{-ft})$-local object and a corresponding $(\mathbb{P}^{1,\delta}, \psi\text{-ft})$-localisation functor $\mathrm{Loc}^{\mathbb{P}^{1,\delta}}_{\psi\text{-ft}}$.

The main theorem of this talk is that there is a morphism $\mathrm{Loc}^{\mathbb{P}^{1,\delta}}_{\psi\text{-ft}} \Sigma^\infty_T \Omega_{\mathfrak{f}} \to \mathbf{\Omega}_{\mathfrak{f}}$ which is a stable $(\mathbb{A}^1, \psi\text{-ft})$-local equivalence. The functors $\mathrm{Loc}^{\mathbb{P}^{1,\delta}}_{\psi\text{-ft}}$ and $\mathrm{Loc}^{\mathbb{A}^1}_{\psi\text{-ft}}$ are unfortunately very inexplicit and this makes this model unusable. The next two talks will be devoted to a more sophisticated version of the same idea, which disentangles the $\mathbb{P}^{1,\delta}$-invariance and $\mathbb{A}^1$-invariance aspects. The references for this talk are an older version [12] of [11] available on demand from the organisers, as well as Chapter 2 of [10].

- Present the material in [12, §2.1]. The most important thing is to explain carefully the construction of compactified exponential connections [12, Construction 2.1.12] and the geometry of the proof of [12, Proposition 2.1.9, Lemma 2.1.13]. The model-category details can be safely ignored. One way to think of this construction is that it is an adaptation and amplification to the foliated context of the "triviality of the complex-analytic Tate twist" $H^2_{\mathrm{dR}}(\mathbb{CP}^1/\mathbb{C}) \simeq \mathbb{C}$ proved using the standard covering of $\mathbb{P}^1$ and the exponential function.
- Explain the construction of localisation functors from §2.6 up to 2.6.24. See [10, Chapter 2] for an overview.

2.3 $\overline{\mathbb{E}}$-localisation II

We need a variant of the construction of the previous talk where the abstract $(\mathbb{P}^{1,\delta}, \psi\text{-ft})$-replacement functor is replaced by a more explicit construction which is enough to carry out the proofs. This new construction involves some relatively complicated diagrams of schemes, which are going to recur throughout the whole week and which should be carefully discussed.

- Introduce the formalism of rigidifications from §2.1. A rigidification of a $S/\mathcal{E}$-foliation $X/\mathcal{F}$ is a diff-smooth morphism $X/\mathcal{F} \to (\mathbb{P}^1)^a \times_{S/\mathcal{E}} \mathbb{G}^b_{\mathrm{m}} \times_{S/\mathcal{E}} (\mathbb{A}^1)^c$, where we keep track of some combinatorial data via a category Θ_+. Rigidifications provide functorial modifications of the foliation on $X/\mathcal{F}$, using compactified exponential foliations associated to closed 1-forms on $S/\mathcal{E}$ (see 2.1.21). Rigidifications are only used in this talk, and do not feature in the model obtained at the end (thanks to Construction 2.7.4, where we keep only the subcategory $\overline{\Theta} \subset \Theta$, corresponding to the $\mathbb{P}^1$-factors), so you can be light on the details here.
- Present the material of §2.2-2.3 around Theorem 2.2.19. This consists in revisiting the constructions of the previous talk, using rigidifications to control the process without doing a ψ-ft localisation.
- Explain the application to $\mathbf{\Omega}_{\mathfrak{f}}$ in §2.4-2.5.

- Describe the $(\overline{\mathbb{E}}, \tau)$-localisation functor from 2.6.25-26. This is at first sight similar to the model of the $(\mathbb{A}^1, \text{ét})$-localisation functor discussed in Talk 1.1; however a major difference between the Suslin-Voevodsky construction and the functor $\underline{\mathrm{Sg}}^{\overline{\mathbb{E}}}$ is that the latter involves a tensor product, which does not a priori preserve topologically fibrant objects.

- Finally, sketch the additional manipulations of diagrams and totalisation functors necessary to arrive at the unital morphism of Theorem 2.7.13.

$$\mathbf{\Omega}_{\mathfrak{f}} \to \mathrm{Tot}^{\mathbb{N},\overline{\Theta}}\, \rho_* \mathrm{R}_{\psi-\mathrm{ft}} \Sigma_T^\infty \widetilde{\mathcal{O}}^\delta.$$

2.4 Foliated motives and motivic Verdier duality

From the formula at the end of Talk 2.3, we see that we have to compute higher direct images of $(T^{\otimes m}) \otimes \widetilde{\mathcal{O}}^\delta$ in the ψ-ft topology. The idea of this talk is that, provided we replace these higher direct images by pushforwards in a category of foliated motives for the ψ-ét topology, we can move the ψ-ft-fibrant replacement out of the pushforwards. This will be an outcome of the six operations formalism applied to foliated motives.

- Reminders on the formalism of stable homotopical 2-functors [7, §1.4.1], with as illustrating example $\mathbf{DA}^{\text{ét}}(-)$ extended from fields to schemes [4, §3]. The most important property for us is Verdier duality: for a smooth projective morphism f, we have a natural isomorphism $f_* \simeq f_\sharp \mathrm{Thom}^{-1}(\Omega_f)$. For an oriented theory like $\mathbf{DA}^{\text{ét}}(-)$, the Thom objects are actually Tate twists [4, Corollary 3.11].

- Explain, following 3.8.1-4 and 3.8.5-13, how one can, under certain assumptions, obtain a full stable homotopical 2-functor from a partial one defined only on smooth schemes.

- Present the construction of the category $\mathbf{FolDA}(S)$ and its variants in §3.1. Since this is very similar to $\mathbf{DA}$ and there are no surprises, you can be quite brief.

- Present the proof of the locality axiom 3.2.16. The proof is very similar to the one of Morel-Voevodsky in motivic homotopy theory; the key difference occurs in step 2 of Proposition 3.2.13, where instead of the Nisnevich local structure of smooth pairs, one uses the ψ-ft-local structure of special closed immersions. This is the motivation for introducing weak completions in the ψ-ft topology, and also justifies thinking about them as substitutes for tubular neighbourhoods.

- Deduce, as in 3.8.5 and 3.8.14, that $\mathbf{FolDA}(-)$ is a stable homotopical 2-functor.

- Present the material in §3.3. Similar arguments have already been discussed in Talk 1.1 for $\mathbf{DA}^{\text{ét}}$ and in Talk 2.3 for the $\overline{\mathbb{E}}$-localisation.

As the ψ-ft topology has infinite cohomological dimension, one cannot apply the method of §3.3 to get an explicit fibrant replacement functor for the ψ-ft cohomology. We develop a workaround by using categories of foliated ind-motives. Because this is a technical point, this should only be sketched and the following is certainly too detailed.

- Present the existence of model structures on categories of ind-objects in certain model categories following §3.4-5.

- Discuss the model categories on ind-complexes of presheaves (§3.6), in particular the hyper-completion step to pass from Definition 3.6.1 to Definition 3.6.3. The key result which allows one to control colimits of ind-objects is Proposition 3.6.6.(b). Explain why, in the ind-context, the arguments of §3.3 go through and give an explicit stable $(\mathbb{A}^1, \tau)$-fibrant replacement even for a topology τ with infinite cohomological dimension (3.6.23).

- Present the material of §3.7 on localisation in the ind-context.

- Explain the application in 3.9; the proof should make it clear why it was important to work in the ind-context (to have a stable $(\mathbb{A}^1, \text{ét})$-localisation functor which commutes with f_*) and why we can still come back to the non-ind context (Proposition 3.6.6.(b)).

- Summarise with the unital morphism of Theorem 3.9.3

$$\boldsymbol{\Omega}_{\mathfrak{f}} \to \underline{\mathrm{Sg}}^{\mathbb{A}^1} \Lambda^{\infty}\, \mathrm{Tot}^{\mathbb{N},\overline{\Theta}}\, \pi_* \mathrm{R}_{\psi-\mathrm{ft}} \rho_* \mathrm{R}_{\psi-\mathrm{Nis}} \Sigma^{\infty}_{T} \widetilde{\mathscr{O}}^{\delta}.$$

3 Wednesday: local study of foliated cohomology, Hodge theory

3.1 Generic hypercoverings and generic foliated cohomology

The motivic spectrum obtained at the end of Talk 2.3 is difficult to understand because it involves a ψ-ft fibrant replacement functor. The ψ-ft topology is rather complicated, and we will develop various computational tools for its cohomology. The task for this talk is to “compute” ψ-ft cohomology at the generic point.

- Discuss the abstract theory of generic hypercoverings from [13, §4.1-4.4]. The setting we have in mind and which should be used for examples is not Δ-schemes with the fttf topology as in [13] but foliations with the ψ-ft topology; the definition in that setting is spelled out in 4.5.4. We also need the similar, simpler case of generic hypercoverings of quasi-compact schemes (as in [13, p. 4.4.2]). In particular, [13, Théorème 4.4.16] should be adapted to theses contexts.

- Present [13, Lemme 4.6.13], simplified to the case of generic hypercoverings of schemes, which is the case which will be used later.

- Present the material in Section 1.8 up to 1.8.11, replacing the ft topology with the ψ-ft topology.

3.2 Local study of foliated cohomology I

To use the morphism of Theorem 3.9.3 to prove something about $\boldsymbol{\Omega}$, it is necessary to compute (or rather construct a unital map from) the restriction of its target motivic spectrum to schemes. Given its definition, this involves computing ψ-ft and ψ-Nis cohomology. In this talk and the next, we eliminate the ψ-ft topology and reduce to a formula involving only ψ-Nis and ψ-ét cohomology. The goal of this talk is to identify a class of foliated presheaves, called malleable presheaves, for which ψ-ét and ψ-ft cohomology of certain foliations coincide (4.6.1), and to develop criteria for malleability (4.6.4). Very roughly speaking, malleable presheaves do not “see” higher differential Galois cohomology in the sense of [13], in the same way that coherent sheaves do not “see” Galois cohomology. One of the key technical tools is the theory of generic hypercoverings developped in Talk 3.1.

- Present the material on power series rings in §4.1-4.3. The statement 4.3.6 is important for the rest of the talk but its proof is too long to be presented in full. Statements 4.2.6, 4.2.7 are only used to prove 4.3.6 and could be skipped.

- Present §4.4, in particular the definition of malleability (4.4.5) and the main inductive theorem (4.4.8).

- Explain the main results in §4.5. This part is very technical.

- Finally we come to the two main results on malleability which will be used in the next talk: 4.6.1 and 4.6.4. Along the way, you will need a result from [13], namely an Eilenberg-Zilber theorem for robust semi-bicosimplicial objects [13, Proposition 5.1.9]. You will also need the family of points described in 1.7.11.

3.3 Local study of foliated cohomology II

To eliminate the ψ-ft fibrant replacement, the rough idea is to use another variant of the foliated Poincaré lemma backwards, i.e., we find a ψ-ft-fibrant replacement $\mathcal{Y}^{\bullet}$ of $\widetilde{\mathscr{O}}^{\delta}$ (4.10.9) such that a closely related presheaf is malleable (4.10.10). A result from the previous talk (4.6.1) then allows to "compute" the restriction of the motivic spectrum from the end of Talk 2.4 to the subcategory SmExFol/k of excellent smooth foliations, which are informally speaking the foliations obtained from smooth schemes by iterated completions.

- Present §4.7-4.8, which give the first examples of malleable presheaves. The rough idea is that, for a presheaf on Sm/k which satisfies a "formal descent" property, the induced presheaf on smooth foliations is malleable, and that this formal descent property is satisfied by $(\mathbb{A}^1, \text{ét})$-local presheaves (where it follows from localisation for $\mathbf{DA}^{\text{ét}}(-)$) and by $\mathbb{G}_{\mathrm{a}}$ (where it follows from invariance of local cohomology by completion).
- Introduce the formalism of *-schemes and foliations, topologies on them, and the basic morphisms of sites §4.9.1-7.
- Define formal flags, and state the criterion for malleability of generalised direct images (Theorem 4.9.13). Compare with the criterion in Theorem 4.6.4., whose proof this is based on. Try to explain the role of formal flags in the proof.
- Present §4.10. First, one introduces a map of presheaves $\widetilde{\mathcal{O}}^\delta \to \mathcal{Y}^\bullet$ on $P^\star$-foliations which induces a ψ-ft-local equivalence on generalised direct images and where $\mathcal{Y}^\bullet$ is built out of de Rham complexes of certain logarithmic connections; the proof relies on the foliated Poincaré lemma. The complex $\mathcal{Y}^\bullet$ is genuinely defined on $P^\star$-foliations, it is the motivation for introducing generalised direct images. Then, the generalised direct image of $\mathcal{Y}^\bullet$ is proved to be malleable using the criterion of 4.9.13.
- Present the local geometry of excellent and smooth excellent foliations in 4.11.1-8.
- Explain the end of §4.11, and summarise with the unital morphism of Theorem 4.11.17

$$\mathbf{\Omega}_{\mathfrak{f}} \to \underline{\mathrm{Sg}}^{\mathbb{A}^1} \Lambda^\infty \,\mathrm{Tot}^{\mathbb{N},\overline{\Theta}}\, \pi_* \mathrm{R}_{\psi-\text{ét}} p_* \mathrm{R}_{\psi-\mathrm{Nis}^\star} \widetilde{\Sigma}^\infty_T \mathcal{Y}.$$

3.4 Variations of mixed Hodge structures and homotopy limits

In this talk, we assume $k = \mathbb{C}$, which is enough to complete the proof in general as was explained in Talk 1.2. We develop an Hodge-theoretic method to show that the unit morphism to the homotopy limit of the cosimplicial cobar construction of certain algebras A is an isomorphism. This is independent of the story so far and will be applied at the end of the proof. Recall that, if the unit of A has a retraction, the morphism $\check{C}^{\leq n+1}(A) \to \check{C}^{\leq n}(A)$ of truncated cobar constructions is null-homotopic for all $n \in \mathbb{N}$ (6.6.5) and this implies that the unit $\mathbb{1} \to \mathrm{Holim}_\Delta \check{C}^\bullet A$ is an isomorphism; this is the principle which was used by Grothendieck to prove faithfully flat descent for modules.

- Explain, following [11, Introduction], how the classical theory of weights on cohomology and the principle above imply the analogue of Conjecture B for $\Sigma^\infty_T \Omega_{/k}$, namely

$$k(0) \xrightarrow{\sim} \operatorname*{Holim}_{\Delta} \check{C}^\bullet \Sigma^\infty_T \Omega_{/k}.$$

- Explain that, more generally, if there exists $N \in \mathbb{N}$ such that for all $n \in \mathbb{N}$, the morphism $\check{C}^{\leq n+N}(A) \to \check{C}^{\leq n}(A)$ of truncated cobar constructions is null-homotopic, the unit morphism is an isomorphism.
- Present quickly the construction of model categories on presheaves with values in a model category 2-functor in §6.1-6.2. This is very close to the classical story of projective and injective model category structures on complexes of presheaves. Moreover, in the rest of the talk, the proofs can be presented in the case where the functor $\alpha : \mathcal{C} \to \mathsf{Sm}/k$ is reduced to a single scheme. Indeed, all of the geometric and Hodge-theoretic arguments are already present in this special case.
- Recall that the category of integrable connections on a smooth k-variety is equivalent to the category of $\mathcal{O}$-coherent $\mathcal{D}$-modules, and that such $\mathcal{D}$-modules are automatically holonomic.

- Explain that the notion of regular singular integrable connections extends to a more general notion of regular singular holonomic $\mathcal{D}$-modules on smooth k-varieties.
- Present the material in §6.3. In particular, explain Beilinson's theorem on derived categories of holonomic $\mathcal{D}$-modules in the form of 6.3.27 and 6.3.31. Note that Beilinson's original result [14] was stated in the language of perverse sheaves, but the general form of the Riemann-Hilbert correspondence (due to Kashiwara-Mebkhout) allows to guess and prove a statement entirely on the regular singular holonomic $\mathcal{D}$-module side.
- Explain how these results of Beilinson allow to relate composition of morphisms in certain towers of algebraic vector bundles (underlying regular singular connections) to the corresponding towers of $\mathbb{R}$-analytic vector bundles (6.4.2).
- Present the canonical $\mathbb{R}$-analytic Deligne splitting of vector bundles underlying variations of mixed Hodge structures (6.5.14).
- Combine the last two ingredients to prove Theorem 6.6.4, which is the sought-for Hodge-theoretic criterion for computation of homotopy limits.

4 Thursday: h-local study, new model for the de Rham spectrum

4.1 h-local study of foliated cohomology I

The aim of this talk and the next is to eliminate excellent foliations and the ψ-ét-fibrant replacements to finally come back to the world of algebraic geometry. The first step is to move from SmExFol/k with the ψ-Nis topology to the larger category ExFol/k with the ψ-h topology. Then we identify a class of presheaves on ExFol/k, the strongly rigid presheaves, for which the ψ-h cohomology is easier to compute.

- Present how to extend the complex of presheaves $\mathcal{Y}$ to ExFol (see 5.8.1-3).
- Explain that the spectrum $\mathbf{\Omega}_{\mathfrak{f}}$ is almost levelwise h-fibrant: its levelwise Zariski fibrant replacement is levelwise h-fibrant. This is proven in the same way as for $\mathbf{\Omega}$ which was seen in Talk 1.1. Note that, combining 4.11.5 with Temkin's resolution of singularities of quasi-excellent schemes in characteristic 0, we can resolve foliations in ExFol/k by foliations in SmExFol/k. Deduce that the morphism defined in terms of SmExFol/k in Theorem 4.11.17 can be extended to a similar morphism on ExFol/k.
- Present the preliminaries on foliations on valuation rings from §5.1-5.3. A fact which may motivate these constructions is the fact that formally saturated polytraits (which have algebraically closed fraction fields by 5.3.18) give a conservative family of points of the h-topos of a noetherian scheme [26, Proposition 2.2, Corollary 3.8, Theorem 4.1].
- Recall the definition of the ψ-h topology on foliations and explain why its restrictions to ExFol/k has good properties. The main result of §5.4 is a description of a conservative family of points of the ψ-h topology on ExFol/k in terms of foliated valuation rings; this is a foliated analogue of the result of Goodwillie and Lichtenbaum mentioned above.
- Define strongly rigid presheaves (5.5.2) and explain how the computation of the ψ-h cohomology of strongly rigid presheaves on foliated valuation rings reduces to a large extent to the closed point (5.5.5 and 5.5.6).

4.2 h-local study of foliated cohomology II

- As the model in Theorem 4.11.17 involves not only the ψ-Nis topology but its more complicated variant on the ψ-Nis* site of morphisms, we need an h-analog, the ψ-hh* topology. Section 5.6 develops, for the ψ-hh* topology, the analogues of the results of §5.4, and in particular constructs a conservative family of points for the ψ-hh* topology.

- Following §5.7, develop the analogues of strong rigidity and its consequences for the computation of ψ-hh* cohomology.
- Present the results of §5.8, i.e., prove the strong rigidity of the presheaves $\mathcal{Y}^\bullet$ and of a certain quotient $\overline{\mathcal{O}}^\times$ of $\mathcal{O}^\times$ (this quotient is perhaps not very intuitive, one way to understand it is to jump ahead and look at its algebraic interpretation in a special case in 7.4.10).
- Finally, present the results of §5.9 and §5.10. The idea is that, for strongly rigid presheaves on morphisms of excellent foliations, their hh*-cohomology and their ψ-hh* cohomology is not so different. In fact, there is a general construction of formal Godement resolutions such that, for some strongly rigid presheaves F (at least for the ones we are interested in, namely $\mathcal{Y}^\bullet \otimes (\overline{\mathcal{O}}^\times)^{\otimes m}$), the formal Godement resolution $\operatorname{Tot}\Theta F$, which is always ψ-hh*-locally equivalent to F (5.9.25), is such that its hh*-cohomology coincides with its ψ-hh* cohomology (5.10.1). The proof of this relies on the general results of section 5.7 on strongly rigid presheaves, but also on some further rigidity properties specific to $\mathcal{Y}^\bullet \otimes (\overline{\mathcal{O}}^\times)^{\otimes m}$.
- Summarise with the unital morphism of Theorem 5.10.16, which does not refer to foliations:

$$\mathbf{\Omega} \to \underline{\mathrm{Sg}}^{\mathbb{A}^1} \Lambda^\infty \operatorname{Tot}^{\mathbb{N},\overline{\Theta}} \pi_* p_* \mathrm{R}_{\mathrm{hh}^\star} \operatorname{Tot} \Xi\, \overline{\Sigma}^\infty_T \mathcal{Y}$$

4.3 A new model for the de Rham spectrum I

The model of $\mathbf{\Omega}$ obtained at the end of the previous talk still suffers from some defects. To start with, its definition still involves weak completions, hence schemes which are not of finite type over k. The goal of this talk is to remedy this situation by "replacing" the weak completions by finite type schemes. The first step is to show that, instead of arbitrary flags, one can without losing anything restrict to flags which are constructible with respect to a strict normal crossings divisor in a smooth variety (7.3.21-22). The formal Godement resolution of $\mathcal{Y}^\bullet$, restricted to such regularly stratified flags, a priori involves some iterated weak completions, but it turns out that these weak completions can be replaced by certain finite type schemes without changing the resolution, up to an hh*-local equivalence (7.4.6).

- Introduce stratified and regularly stratified schemes (7.1.1-4).
- Explain, following 7.3, that resolution of singularities allows to reformulate the model obtained at the end of Talk 4.2 in terms of regularly stratified schemes (7.3.22).
- Present the construction of schemes attached to stratas in regularly stratified schemes from 7.1 (from 7.1.5 on). These constructions will accompany us till the end of the proof and need to be explained carefully.
- Explain, following 7.2, the structure and functoriality of integrable connections on regularly stratified schemes. This uses the de Rham nearby cycles functor in this special case.
- Explain how to use these construction to "algebraise" the model obtained earlier in the talk (§7.4). The key new ingredient to perform this algebraisation is the theorem of Beilinson on the relationship between de Rham cohomology and Ext groups in the category of $\mathcal{D}$-modules [15, Lemma 2.1.1], which was already used in a different way in Talk 3.4.
- Summarise with the unital morphism of Theorem 7.4.22

$$\mathbf{\Omega} \to \underline{\mathrm{Sg}}^{\mathbb{A}^1} \Lambda^\infty \operatorname{Tot}^{\mathbb{N},\overline{\Theta}} \pi_* \tilde{p}_* g_* \mathrm{R}_{\mathrm{hh}^\star} \operatorname{Tot}^\sharp \Xi((\widetilde{\mathcal{O}}^\times)^{\otimes m} \otimes R\Gamma^\delta \underline{\mathcal{Y}})_m.$$

4.4 A new model for the de Rham spectrum II

The model from the end of the last talk is already very close to what is needed for the end of the proof. However, it still requires a small modification, to reveal the existence of a certain double filtration in terms of dimensions of the schemes in the diagram. Moreover, it is possible to push the formal Godement resolution into the suspension spectrum.

- Present the results of §7.5. The key technical result is a base change theorem for logarithmic connections (7.5.28).
- Explain the filtration by dimension from §7.6-7.7. Emphasise the concrete situation we are interested in rather than the categorical generalities (whose proofs could be skipped altogether).
- Present the results of §7.8. Many arguments in this section are inspired by classical results in the study of Voevodsky motives, and in particular the computation of the weight 1 motivic complex [34, Lecture 4].
- Summarise the results of §7.5, §7.8 with the unital morphism of Theorem 7.5.36, modified using §7.8,
$$\boldsymbol{\Omega} \to \underline{\mathrm{Sg}}^{\mathbb{A}^1} \Lambda^\infty \, \mathrm{Tot}^{\mathbb{N},\overline{\Theta}} \, \pi_* \tilde{p}_* g_* \mathrm{R}_{\mathrm{hhh}\clubsuit} \Sigma^\infty_T \, \mathrm{Tot}^\sharp \, \Xi \, \mathscr{X}.$$

5 Friday: Moving by blow-ups and applications, end of the proof

5.1 Moving by blow-ups and the Bloch-Levine tower

The end of the proof relies crucially on an extension of Bloch's technique of "moving by blow-ups" [16]. Bloch's theorem states that, given a normal crossing divisor D in a smooth variety X over a field and an arbitrary irreducible subscheme Z of codimension at least r, it is possible to find an iterated blow-up of X at faces of D such that the strict transform of Z meets every face of the induced normal crossing divisor of the blow-up in codimension at least r. This was used in [16] to prove the localisation property for Bloch's higher Chow groups for smooth varieties over a field, and was extended by Levine [31] [32] [33] to prove localisation for more general theories and smooth schemes over a semi-local principal ideal domain (e.g. a discrete valuation ring).

The first aim of this talk is to present an extension of this technique which works over arbitrary valuation rings; this is useful because we are systematically working with the h-topology and want to prove that certain maps are h-local equivalences. The method of proof is different from the one of Bloch and Levine: their combinatorial arguments are replaced by a compactness argument, using the description of the pro-finite space of vertices of the pro blow-up of all faces as a space of total orderings.

- Present the results from §8.1 up to 8.1.15, i.e., the new proof of Bloch's original theorem. To simplify things, instead of introducing directly the notions of toroidaly stratified schemes, you should explain the statement and proof in the case of a regularly stratified scheme.
- Introduce toroidaly stratified schemes. Give some examples of such schemes which are not regularly stratified.
- Present the results of 8.1.16-26 on the structure of toroidaly stratified schemes over a polytrait.
- Explain how to adapt the moving by blow-up to the polytrait setting. You should try to rely on the proof of the simpler case above and just point out the main complications in the combinatorics in Lemma 8.1.29 compared to Lemma 8.1.13.

The second is to introduce a first version of the Bloch-Levine tower, namely the effective Bloch-Levine tower for schemes, and to motivate it by recalling a theorem of Levine on the slice filtration.

- Present the variant $\Pi(F)$ of the Suslin-Voevodsky construction, its Bloch-style version $\widehat{\Pi}(F)$ using admissible blow-ups, and the result that they coincide for h-fibrant presheaves (§8.4.1-8).
- Define the Bloch-Levine tower (§8.4.9-19). Note that the functoriality of the Levine tower for general morphisms over perfect fields in [33] relies on the Chow moving lemma; here, the problem is circumscribed using the notion of levelled schemes and levelled morphisms (which generalize flat morphisms).

- (Time permitting) Define the slice tower of a complex of presheaves on smooth varieties [33, §7.1]. Combine [33, Theorem 7.11] and 8.4.8 and state that the Bloch-Levine tower of π^*K for K an $(\mathbb{A}^1, \text{ét})$-local complex K of presheaves of k-vector spaces on Sm/k models the slice tower of K. One has to be careful that the Bloch-Levine tower we consider receives a map from what Levine denotes by $E^{(0/p)}$ and not from $E^{(p)}$.

  This result on the slice tower and its analogue for T-spectra are not directly used in the end of the proof but they are relevant because, as we have seen in Talk 3.4, we want to control the defect of $\mathbb{P}^1$-effectivity of $\mathbf{\Omega}$, and also of $\text{Holim}_\Delta \check{C}^\bullet(\mathbf{\Omega})$ which we expect to be $\mathbb{P}^1$-effective by Conjecture B. Also, we should point that an important ingredient in the proof of [33, Theorem 7.11] is localisation for the homotopy coniveau tower [33, Theorem 3.2.1], which is proven using moving by blow-ups, so we are quite close to Levine's setting.

5.2 Contractions and $(\mathbb{A}^1, h)$-equivalences

The main result of this talk is that the étale-local version of Voevodsky's contraction preserves $(\mathbb{A}^1, \text{h})$-local equivalences of complexes of presheaves of $\mathbb{Q}$-vector spaces. Over a field of characteristic 0, such $(\mathbb{A}^1, \text{h})$-local equivalences are known to coincide with $(\mathbb{A}^1, \text{ét})$-local equivalences; this result is generalised in the second part of this talk. The preservation of $(\mathbb{A}^1, \text{ét})$-local equivalences by contraction is then a classical property of contraction, following for instance from the comparison $\mathbf{DA}^{\text{ét,eff}}(k, k) \simeq \mathbf{DM}^{\text{eff}}(k, k)$ and the general results of Voevodsky on homotopy invariants presheaves with transfers (see [34, Chapter 23,24]).

In general, one has to understand the case of a polytrait and argue by induction on the length. The proof is obtained by introducing variants of the h and étale topologies on schemes over a polytrait, the so-called rig-h and rig-ét topologies, which are generated by the h and étale topologies together with admissible modifications in the sense of Raynaud. The idea is then to separate the proof in two: first, show that contraction sends $(\mathbb{A}^1, \text{rig-ét})$-local equivalences to $(\mathbb{A}^1, \text{rig-h})$-local equivalences, and then show that $(\mathbb{A}^1, \text{rig-h})$-local equivalences coincide with $(\mathbb{A}^1, \text{rig-ét})$-local equivalences.

- Present the results of §8.2. The idea is to construct the relevant $(\mathbb{A}^1, \text{rig-h})$-local equivalences by controlling the necessary admissible modifications in terms of toroidal modifications. This involves using moving by blow-ups arguments to prove a kind of semi-stable reduction theorem relative to a polytrait (Proposition 8.2.19). Unfortunately, one cannot apply directly the moving results of the previous talk and one has to revisit some of the constructions.

- Present the results of §8.3. This uses an adaptation to the rig-topologies of some standard devissages, due to Voevodsky, of the h-topology into the cdh-topology (for which one can use the theory of cd-structures) and Galois coverings.

- Explain how to use Theorem 8.2.1 to modify the morphism from the end of Talk 4.4 to get a unital morphism

$$\mathbf{\Omega} \to \underline{\text{Sg}}^{\mathbb{A}^1} \, \text{Tot}^{\mathbb{N},\overline{\Theta}} \, \pi_* \tilde{p}_* \text{R}_{\text{hhh}^\clubsuit} \widetilde{\Lambda}^\infty \text{R}_{\text{ét}^\clubsuit} \Sigma_T^\infty g_* \text{R}_{\text{hhh}^\clubsuit} \, \text{Tot}^\sharp \, \Xi \, \mathscr{Z}.$$

5.3 Stable Bloch-Levine tower and dimension bounds

The goal is to extend the construction of Bloch-Levine towers to T-spectra and to the context of $\clubsuit$-schemes, and to prove a key result which allows to compare two stabilised $\text{hhh}^\clubsuit$-local Bloch-Levine towers at a fixed level p by comparing sections on $k^\clubsuit$-polytraits of "small dimension" depending on p in a suitable sense.

- Explain the simple extension of the Bloch-Levine tower to T-spectra, still in the scheme context (8.4.20-23).

- Explain how the stabilised, h-local version of the Bloch-Levine tower of T-spectra admits a "quotient" where we also consider a colimit over open subsets in $\mathbb{G}_\text{m}^r$ with controlled codimension of the complement (end of §8.4).

- Explain, following §8.5, how to extend the construction of the Bloch-Levine tower and its (unstabilised and stabilised) versions to $k^{\clubsuit}$-schemes, replacing at the same time the diagram of affine spaces $(\mathbb{A}(I))_I$ going into the Bloch tower by the diagram $(P^{\clubsuit}(I), \overline{\Theta}^{(\omega^e(I))})_{e,I}$ and incorporating $\mathbb{E}$-localisation.

- Present the results of §8.6, which show how a given level of the Bloch-Levine tower in its $\mathbf{BL}^{\clubsuit}$ version "only depends" on its sections on polytraits of controlled dimensions. This relies in a crucial way on the moving by blow up results of Talk 5.1.

5.4 End of the proof

The End is Nigh! We compute the stabilised Bloch-Levine tower of the Čech cobar construction of the model of $\mathbf{\Omega}$ obtained at the end of Talk 5.2. To be in a position to apply the Hodge-theoretic main result of Talk 3.4, we use the dimensional filtration of the new model of $\mathbf{\Omega}$ and the main result on dimension bounds in the stable Bloch-Levine tower of Talk 5.3.

- Recapitulate the setup of the end of the proof as in §8.7.

- Explain the key step of the computation in §8.8.

- Present the results of §8.9. The aim here is to fold back the Bloch-Levine tower part of the construction. The proof relies on classical results on motives and motivic homotopy theory: vanishing of motivic cohomology and Morel-Voevodsky localisation.

- Present the results of §8.10, which simplify the obtained spectrum by removing the machinery of $\clubsuit$-schemes and of flags, and recast the end result in the language of excellent foliations and the ψ-ét topology. This finally gives a unital quotient of the homotopy limit which is (-1)-connected and ends the proof of Theorem D.

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