J-holomorphic curves in symplectic geometry

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Since their introduction by Gromov [4] in the mid-1980's J-holomorphic curves have been one of the most widely used tools in symplectic geometry, leading to the formulation of various theories (Gromov-Witten invariants, quantum cohomology, various Floer homologies, symplectic field theory, Fukaya category), answers to old questions in symplectic geometry (various Arnold conjectures) and the discovery of new phenomena (e.g. non-squeezing). It was the modest aim of these lectures to explain some of the very basic underlying principles and techniques and illustrate their use in the study of Lagrange embeddings.

The first lecture started with a brief introduction to symplectic geometry. I mentioned some of the typical (very general) questions, such as the

Existence problem: Given an compact almost complex manifold (M, J) with a class $a \in H^2(M; \mathbb{R})$ such that $a \cup \cdots \cup a \neq 0 \in H^{2n}(M; \mathbb{R})$, does M admit a symplectic form representing this class (and which tames some almost complex structure homotopic to J)?

and the

Mapping problem: How special are symplectomorphisms as opposed to just volume-preserving diffeomorphisms? Can one give lower bounds on the number of fixed points of Hamiltonian diffeomorphisms in terms of the topology of M?

The second half of the lecture consisted of a discussion of Lagrangian immersions and embeddings of compact manifolds into standard $(\mathbb{R}^{2n} \cong \mathbb{C}^n, \omega = \sum dx_k \wedge dy_k)$ and what one can say without the use of holomorphic curves. I stated the Gromov-Lees theorem asserting that Lagrangian immersions into $\mathbb{C}^n$ satisfy the h-principle, so that regular homotopy classes of Lagrange immersions are in bijective correspondence with homotopy classes of U-parallelizations of L. I also mentioned the Whitney immersions of spheres with a single double point. This was followed by a brief illustration of Givental's construction of immersions of surfaces. I also sketched the proof of the fact that the product of a Lagrangian immersion $f : V^n \to \mathbb{C}^n$ with a Lagrangian embedding $g : W^m \to \mathbb{C}^m$ $(m \geq 1)$ is homotopic in the class of Lagrangian immersions to a Lagrangian embedding of $V \times W$ into $\mathbb{C}^{n+m}$. I next discussed how one can remove transverse double points by Lagrange surgery. Using Whitney's algebraic count of the number of double points of an immersion in terms of the Euler class of the normal bundle and the isomorphism $TL \cong NL$ for immersed Lagrangian submanifolds, we saw that the Lagrangian embeddability question can be decided completely for all surfaces except the Klein bottle. Similarly, for all spheres except S^3 one can decide this question purely in terms of algebraic topology, with the result the only sphere (with the possible exception of S^3) that admits a Lagrangian embedding into $\mathbb{C}^n$ is the circle S^1. On the other hand, whereas all oriented closed 3-manifolds are parallelizable and so admit Lagrangian immersions into $\mathbb{C}^3$, there are no "classical" obstructions to the existence of Lagrangian embeddings.

At the end I stated Gromov's result that there are no exact Lagrangian embeddings into $\mathbb{C}^n$, as well as the non-squeezing theorem and the C^0-closedness of the symplectomorphism group to give some of the motivations for the study of holomorphic curves. Good references for the material covered in this lecture are the books [5] and [1] (here especially chapter X).

The second lecture started with the basics on almost complex structures on symplectic manifolds and J-holomorphic curves, in particular with a proof that the energy of a J-holomorphic curve for some tamed J is given by the pull-back of the symplectic form. I also mentioned that for compatible J, the symplectic form is a calibration, so that J-curves are conformal parametrizations of absolutely area-minimizing surfaces. After a very sketchy discussion of how one sets up the proof that for generic J the space of simple J-holomorphic spheres in a given class $A \in H_2(M;\mathbb{R})$ is a manifold whose oriented cobordism class is independent of the specific regular J, I stated the basic compactness result under the assumption of L_∞-bounds. This was followed by a description of the process of *bubbling*, which I illustrated with the standard example of a family of smooth quadrics in $\mathbb{C}P^2$ "converging to" a pair of lines. Next I discussed the important special case of homology classes with minimal positive symplectic area, for which one gets compactness of the moduli spaces of maps up to reparametrization. I used this absence of bubbling for the class $[S^2 \times pt] \in H_2(S^2 \times \mathbb{C}^{n-1})$ to discuss the proof of Gromov's

Non-squeezing Theorem.[4] *If $B^{2n}(0,R)$ embeds symplectically into $B^2(0,r) \times \mathbb{C}^{n-1}$ then $R \leq r$.*

An excellent reference for the material of the second lecture is the book [6] (in particular chapters 2–4).

The third lecture was dedicated to a discussion of Lagrangian embeddings into $\mathbb{C}^n$ using J-holomorphic curves. I started by proving Gromov's

Theorem.[4] *Suppose $L \subset \mathbb{C}^n$ is an embedded compact Lagrangian submanifold. Then there exists a nonconstant holomorphic map (for the standard complex structure on $\mathbb{C}^n$) $u : (D^2, \partial D^2) \to (\mathbb{C}^n, L^n)$.*

Here the basic line of argument is that compactness for a suitable 1-parameter family of perturbed Cauchy-Riemann equations

$$\bar{\partial} u = a(t)$$

has to fail for topological reasons, resulting in the existence of a bubble. As the limiting map is obtained via rescaling, it satisfies the homogeneous equation, i.e. it is holomorphic. Moreover, this bubble must be a disc, since in an exact symplectic manifold such as $\mathbb{C}^n$ there are no non-constant holomorphic spheres. Of course, the theorem implies that $H_1(L;\mathbb{R}) \neq 0$, proving that S^3 does not admit a Lagrangian embedding into $\mathbb{C}^3$.

Up to this point the proofs involved only the very crude dichotomy "compactness or bubbling". Next I gave a pictorial description of what the general limit of a sequence of holomorphic discs in a fixed homology class looks like, with the assertion that interior bubbling adds 2 to the codimension and boundary bubbling adds 1, so that the boundary (in the sense of algebraic topology) of a moduli space of discs can be described by pairs of discs meeting at a boundary point. After a brief discussion of the Maslov index and the statement of the Audin conjecture, I formulated Fukaya's

Theorem.[3] *Suppose L^n is an embedded Lagrangian submanifold in $\mathbb{C}^n$ which is spin and aspherical. Then there exists a map $u : (D^2, \partial D^2) \to (\mathbb{C}^n, L)$ with the following properties:*

(i) the symplectic area satisfies $E(u) > 0$,

(ii) the Maslov index of the boundary loop satisfies $\mu(u_{|\partial D}) = 2$, and

(iii) the loop $\gamma = u_{|\partial D}$ has centralizer $Z_\gamma = \{a \in \pi_1(L) : a\gamma = \gamma a\}$ of finite index in $\pi_1(L)$.

and its

Corollary.[3] *An oriented closed prime 3-manifold L^3 admits a Lagrangian embedding into $\mathbb{C}^3$ if and only if L is diffeomorphic to $S^1 \times \Sigma$ for some closed oriented surface Σ.*

To give an idea of the proof of the theorem, I next introduced the relevant parts of string topology in the sense of Chas and Sullivan [2]. I gave the definition of the string bracket on the transversal chain level and illustrated it with the example of two orthogonal families of closed loops on a two-dimensional torus. Then I asserted that given a Lagrangian submanifold $L \subset (M, \omega)$ of a symplectic manifold, one can view the moduli space $\mathcal{M}(A; J)$ of holomorphic discs in a given relative homology class $A \in H_2(M, L)$ as a chain in the closed string space ΣL by assigning to (an equivalence class of) discs its (arc-length parametrized) boundary data. The formal sum

$$c := \sum_{A \in H_2(M,L)} c_A z^A \in C_*(\Sigma L)$$

of these chains then satisfies *Fukaya's first equation*

$$\partial c + \frac{1}{2}[c, c] = 0,$$

where $[\,.\,,\,.\,]$ denotes the string bracket. Again I gave a quick illustration of this phenomenon for the class $A = (1,1) \in H_1(T^2) \cong H_2(\mathbb{C}^2, T^2)$ for the standard torus $T^2 = S^1 \times S^1 \in \mathbb{C} \times \mathbb{C}$. I ended by indicating that to prove Fukaya's theorem, one now again looks at the Gromov trick of using the perturbed Cauchy-Riemann equation

$$\bar{\partial} u = a(t)$$

for a suitable family of perturbations $a : [0,1] \to \mathbb{C}^n$ starting at $a(0) = 0$ and ending at some $a(1)$ whose norm is so large that there are no solutions. Assuming sufficient regularity, for all but a finite number of $t \in [0,1]$ the moduli space of solutions for right hand side $a(t)$ will be essentially a manifold, giving rise to a chain $\sigma_t \in C_*(\Sigma L)$ as before. We also denote by $\sigma \in C_*(\Sigma L)$ the chain of one higher dimension which is obtained from the whole family. Then σ satisfies *Fukaya's second equation*

$$\partial \sigma = \sigma_0 \pm [c, \sigma],$$

where as before c is the formal series describing the holomorphic discs with boundary on L and the bracket is the string bracket. Restricting to the class $A = 0$ in this equation yields

$$(\partial \sigma)_{A=0} = (\sigma_0)_{A=0} - \sum_{B \in H_2(\mathbb{C}^n, L)} [c_B, (\sigma)_{-B}].$$

It follows from the dimension formulas

$$\begin{aligned} \dim c_A &= n - 3 + \mu(A) \\ \dim \sigma_{-B} &= n + 1 - \mu(B) \end{aligned}$$

that if $c_A = 0$ for all classes with $\mu(A) = 2$, then in all appearing bracket expressions at least one of the chains c_B, σ_{-B} has dimension < 0 (i.e. it is empty) or $> n$. In the theorem we assume that L is aspherical, so $H_k(\Sigma L) = 0$ for $k > n$. The idea is now that one can inductively eliminate all the bracket terms (even though the appearing terms are not all cycles!!), finally showing that $(\sigma_0)_{A=0}$ was a boundary for the ordinary boundary operator ∂ on $C_*(\Sigma L)$. This is the desired contradiction since the inclusion of constant strings into the component of null-homotopic strings is a homotopy equivalence in the case at hand. Thus the assumption that $c_A = 0$ for all classes with $\mu(A) = 2$ was wrong.

It might be interesting to note that the inductive procedure can be interpreted as a sequence of "gauge transformations" for the "flat connection" c.

References

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