Stochastik II
Wintersemester 2018/19
Humboldt-Universität zu Berlin
Prof. Dr. Dörte Kreher

Exercise Sheet 1

Problem 1 (4 Points)
Let X and Y be independent, identically distributed random variables with

$$\mathbb{P}(X=1)=\mathbb{P}(Y=1)=p\in(0,1), \qquad \mathbb{P}(X=-1)=\mathbb{P}(Y=-1)=1-p.$$

We set $Z:=\mathbb{1}_{\{X+Y=0\}}$ and $\mathcal{G}:=\sigma(Z)$. Compute $\mathbb{E}(X|\mathcal{G})$ and $\mathbb{E}(Y|\mathcal{G})$. Are these random variables still independent?

Problem 2 (4 Points)
Let $X\in L^2(\Omega,\mathcal{F},\mathbb{P})$ and let $\mathcal{G}\subset\mathcal{F}$ be a sub-σ-algebra of $\mathcal{F}$.

a) We define the conditional variance $\mathrm{Var}(X|\mathcal{G}):=\mathbb{E}((X-\mathbb{E}(X|\mathcal{G}))^2|\mathcal{G})$. Show that

$$\mathrm{Var}(X)=\mathbb{E}(\mathrm{Var}(X|\mathcal{G}))+\mathrm{Var}(\mathbb{E}(X|\mathcal{G})).$$

b) Let Y be another random variable on $(\Omega,\mathcal{F},\mathbb{P})$ and suppose that $\mathbb{E}(X^2|Y)=Y^2$ a.s. and $\mathbb{E}(X|Y)=Y$ a.s. Show that $X=Y$ a.s.

Problem 3 (4 Points)
Let $(\Omega,\mathcal{F},\mathbb{P})$ be a probability space, $\mathcal{G},\mathcal{H}$ sub-σ-algebras of $\mathcal{F}$ and $X\in L^1(\Omega,\mathcal{F},\mathbb{P})$ or $X\geq 0$.

a) Show: If $\mathcal{H}\subset\mathcal{G}\subset\mathcal{F}$, then $\mathbb{E}(\mathbb{E}(X|\mathcal{G})|\mathcal{H})=\mathbb{E}(\mathbb{E}(X|\mathcal{H})|\mathcal{G})=\mathbb{E}(X|\mathcal{H})$ a.s.

b) Give an example where $\mathbb{E}(\mathbb{E}(X|\mathcal{G})|\mathcal{H})\neq\mathbb{E}(\mathbb{E}(X|\mathcal{H})|\mathcal{G})$.

c) Show: If $\mathcal{H}$ is independent of $\sigma(\sigma(X),\mathcal{G})$, then $\mathbb{E}(X|\sigma(\mathcal{G},\mathcal{H}))=\mathbb{E}(X|\mathcal{G})$ a.s.

Problem 4 (4 Points)
Let $(X_n)_{n\in\mathbb{N}}\subset L^1(\Omega,\mathcal{F},\mathbb{P})$ be independent and identically distributed. For each $n\in\mathbb{N}$ we set $S_n:=\sum_{i=1}^n X_i$ and $\mathcal{G}_n:=\sigma(S_n,S_{n+1},\dots)$. Show that:

a) $\mathbb{E}(X_j|\mathcal{G}_n)=\mathbb{E}(X_j|S_n)$ a.s. for all $1\leq j\leq n$,

b) $\mathbb{E}(X_j|\mathcal{G}_n)=\mathbb{E}(X_1|S_n)$ a.s. for all $1\leq j\leq n$,

c) $\mathbb{E}(X_j|\mathcal{G}_n)=\frac{S_n}{n}$ a.s. for all $1\leq j\leq n$.

Please hand in your solutions on Monday, 29 October, 2018 at 9:15 am.