Stochastik II
Wintersemester 2018/19
Humboldt-Universität zu Berlin
Prof. Dr. Dörte Kreher

Exercise Sheet 10

Problem 1 (4 Points)
The proof copy of a book is read multiple times by the editor checking for mistakes. Each mistake is detected with probability $p \in (0,1)$ at each reading, independently of the others. Between readings the printer corrects the detected mistakes, but introduces an independent Poisson(λ)-distributed number of new mistakes.

a) Find a recursive expression for the function $\varphi_n(s) := \mathbb{E}(s^{X_n})$ with $s \in [0,1]$, where X_n denotes the number of mistakes after the n-th editor-printer cycle, and determine its limit as $n \to \infty$.

b) Show that $X = (X_n)_{n\in\mathbb{N}_0}$ is a reversible Markov chain with invariant distribution $\pi = \text{Poisson}(\lambda/p)$.

Problem 2 (4 Points)
Let X be an irreducible, positive recurrent, aperiodic Markov chain with transition matrix $P = (p_{ij})_{i,j\in E}$. Show that X is reversible if and only if for all $n \in \mathbb{N}$ and all $j_1, \ldots, j_n \in E$,

$$p_{j_1j_2}p_{j_2j_3}\cdots p_{j_{n-1}j_n}p_{j_nj_1} = p_{j_1j_n}p_{j_nj_{n-1}}\cdots p_{j_2j_1}.$$

Problem 3 (4 Points)
Let $(Y_n)_{n\in\mathbb{N}}$ be independent, identically distributed $\mathbb{N}$-valued random variables with

$$\gcd\{n \in \mathbb{N}:\ \mathbb{P}(Y_1 = n) > 0\} = 1 \quad \text{and} \quad \mu := \mathbb{E}(Y_1) < \infty.$$

Define $X_0 := 0$ and for each $n \in \mathbb{N}$,

$$X_n := \inf\{m \geq n:\ m = Y_1 + \cdots + Y_k \text{ for some } k \geq 0\} - n.$$

Show that

$$\mathbb{P}(X_n = 0) = \mathbb{P}(n = Y_1 + \cdots + Y_k \text{ for some } k \geq 0) \to \frac{1}{\mu} \quad \text{as} \quad n \to \infty.$$

Problem 4 (4 Points)
A fair die is thrown repeatedly. For each $n \in \mathbb{N}$ let X_n denote the sum of the first n throws. Show that $\lim_{n\to\infty} \mathbb{P}(X_n \text{ is a multiple of } 13)$ exists and determine its value.

Please hand in your solutions on Monday, 14 January, 2019 at 9:15 am.