Stochastik II
Wintersemester 2018/19
Humboldt-Universität zu Berlin
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Exercise Sheet 11

Problem 1 (5 Points)
Let X be a Markov chain on $\{0,1,2,3,4,5,6\}$ with transition matrix

$$P = \begin{pmatrix} \frac{1}{5} & \frac{3}{5} & 0 & 0 & \frac{1}{5} & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & \frac{2}{3} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

a) Compute for X
   (i) starting from 0, the probability of hitting 6,
   (ii) starting from 1, the average number of steps it takes to hit 3,
   (iii) starting from 1, the long run proportion of time spent in 2.

b) Do the limits of $p_{01}^{(n)}$ and $p_{04}^{(n)}$ exist as $n \to \infty$? If yes, compute the respective limit.

Problem 2 (3 Points)
A professor has N umbrellas. He walks to the office in the morning and walks home in the evening. If it is raining he likes to carry an umbrella and if it is fine he does not. Suppose that it rains on each journey with probability p, independently of the past. What is the long-run proportion of journeys on which the professor gets wet? Assume for simplicity that it does not start raining in the middle of any journey, i.e. either it is raining already at the beginning of his journey or it does not rain at all during his journey.

Problem 3 (4 Points)
Let $X = (X_n)_{n\in\mathbb{N}_0}$ be a Markov chain on a countable state space E with transition matrix $P = (p_{ij})_{i,j\in E}$. We call a function $u : E \to [0,\infty)$ *superharmonic* if

$$u(i) \geq \sum_{j\in E} p_{ij} u(j) \quad \text{for all } i \in E.$$

Show that:

a) $(u(X_n))_{n\in\mathbb{N}_0}$ is a $\mathbb{P}_i$-supermartingale for any $i \in E$.

b) $\mathbb{E}_i(u(X_T); T < \infty) \leq u(i)$ for any $i \in E$ and any stopping time T.

c) If X is irreducible and recurrent, then any superharmonic function $u : E \to [0,\infty)$ is constant on E.

Problem 4 (4 Points)
Consider the simple voter model where N people vote in each period independently of each other either for "1" or for "0". If $x = (x_1, \dots, x_N) \in E := \{0,1\}^N$ denotes the present configuration and

$$m(x) := \frac{1}{N} \sum_{i=1}^{N} x_i$$

defines the present "sentiment" of the crowd, then in the next period the i-th person will vote for "1" with probability

$$p_i(x) = \alpha x_i + (1-\alpha) m(x) \quad \text{for some given } \alpha \in (0,1), \quad i = 1, \dots, N.$$

a) Describe this evolution as a Markov chain $(X_n)_{n \in \mathbb{N}_0}$ on the state space E.

b) Show that $(m(X_n))_{n \in \mathbb{N}_0}$ is a martingale.

c) Prove that in the limit as $n \to \infty$ the process is absorbed with probability $q_0 := \mathbb{E}(m(X_0))$ in the state $\{1, 1, \dots, 1\}$ (all vote for "1") and with probability $1-q_0$ in the state $\{0, 0, \dots, 0\}$ (all vote for "0").