Stochastik II
Wintersemester 2018/19
Humboldt-Universität zu Berlin
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Exercise Sheet 12

Problem 1 (4 Points)
Let $(E, \mathcal{E})$ be a Polish space and let T be an arbitrary index set. For every $t \in T$ let $\mathbb{P}_t$ be a probability measure on $(E, \mathcal{E})$ and let $P_J := \bigotimes_{t \in J} \mathbb{P}_t$ be the product measure of the $\mathbb{P}_t$, $t \in J$, for any finite subset $J \subset T$. Then $\mathbb{P}_J$ is a probability measure on $(E^J, \mathcal{E}^J)$ for every finite subset $J \subset T$. Prove that there exists a unique probability measure $\mathbb{P}$ on $(E^T, \mathcal{E}^T)$ such that $\mathbb{P}_J = \mathbb{P} \circ (\pi_J)^{-1}$ for every finite subset $J \subset T$ and that the $(\pi_i)_{i \in T}$ are independent under $\mathbb{P}$.

Problem 2 (4 Points)
Let $(\tau_k)_{k \in \mathbb{N}}$ be a sequence of independent, identically exponentially distributed random variables with parameter $\lambda > 0$. Define for every $n \geq 1$,

$$\sigma_n := \sum_{k=1}^{n} \tau_k, \qquad S_n := (\sigma_1, \ldots, \sigma_n).$$

Consider the Poisson process

$$N_t := \sum_{k=1}^{\infty} \mathbb{1}_{[0,t]}(\sigma_k), \quad t \geq 0.$$

a) Prove that for every $t \geq 0$, $N_t \sim \text{Poisson}(\lambda t)$.

b) Prove that for every $t \geq 0$, $n \in \mathbb{N}$, and $B \in \mathcal{B}(\mathbb{R}^n)$,

$$\mathbb{P}(S_n \in B | N_t = n) = \frac{n!}{t^n} \int_B \mathbb{1}_{\Delta_t^{(n)}}(t_1, \ldots, t_n) dt_1 \ldots dt_n,$$

where $\Delta_t^{(n)} := \{(t_1, \ldots, t_n) \in \mathbb{R}^n : \ 0 < t_1 < \cdots < t_n < t\}$.

Problem 3 (4 Points)
Consider the Poisson process $N = (N_t)_{t \geq 0}$ defined in Exercise 2. Show that N is $\mathbb{P}$-almost surely continuous at each point $t > 0$. Conclude that N is continuous in probability, but does not have continuous trajectories. In fact, $\mathbb{P}$-almost surely N has infinitely many discontinuities.

Problem 4 (4 Points)
Let $B = (B_t)_{t \geq 0}$ be a standard Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Show that:

a) The moments of a standard Gaussian random variable $Z \sim N(0, 1)$ are given by

$$\mathbb{E}(Z^k) = \begin{cases} \frac{2^{-k/2} k!}{(k/2)!} & k \text{ even} \\ 0 & k \text{ odd} \end{cases}.$$

b) There exists a modification of B with locally Hölder continuous sample paths of any order $\gamma < 1/2$.

Please hand in your solutions on Monday, 28 January, 2019 at 9:15 am.