Stochastik II
Wintersemester 2018/19
Humboldt-Universität zu Berlin
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Exercise Sheet 13

Problem 1 (4 Points)
Let $(B_t)_{t\geq 0}$ be a Brownian motion.

a) Prove that the following processes are also Brownian motions on $[0,\infty)$ resp. $[0,1]$:
- $B^{(1)} = \left(B_t^{(1)}\right)_{t\geq 0}$ defined by $B_t^{(1)} := \frac{1}{a}B_{a^2 t}$, for $t \geq 0$ and for a fixed $a > 0$.
- $B^{(2)} = \left(B_t^{(2)}\right)_{t\geq 0}$ defined by $B_t^{(2)} := tB_{1/t}$ for $t > 0$ and $B_0^{(2)} := 0$.
- $B^{(3)} = \left(B_t^{(3)}\right)_{t\in[0,1]}$ defined by $B_t^{(3)} := B_{1-t} - B_1$ for $0 \leq t \leq 1$.

b) Show that

$$\lim_{t\to\infty} \frac{B_t}{t} = 0 \quad \text{almost surely.}$$

Problem 2 (2 Points)
A centered Gaussian process $X = (X_t)_{t\in[0,1]}$ having almost surely continuous trajectories is called a Brownian bridge if its covariance function $\Gamma : [0,1]\times[0,1] \to \mathbb{R}$ is given by $\Gamma(s,t) = s\wedge t - st$ for $s,t \in [0,1]$. Let $(B_t)_{t\in[0,\infty)}$ be a Brownian motion. Show that the following two processes are Brownian bridges:

a) $X_t^{(1)} := B_t - tB_1,\ t \in [0,1]$.

b) $X_t^{(2)} := (1-t)B_{\frac{t}{1-t}}$ for $t \in [0,1)$ and $X_1^{(2)} := 0$.

Problem 3 (4 Points)
Let $X = (X_t)_{t\geq 0}$ be a real-valued, centered Gaussian process with $\mathrm{Var}(X_t) = 1$ for all $t \geq 0$, defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and suppose that X_t and X_s are independent for all $s \neq t$. Prove that:

a) There does not exist a modification of X having continuous trajectories.

b) There does not exist a $\mathcal{B}([0,1]) \otimes \mathcal{F}$ - measurable modification of X.

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Problem 4 (6 Points)
Let $(X_t)_{t\in[0,1]}$ be a real-valued, centered Gaussian process on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with covariance function K and suppose that the map $(t, \omega) \mapsto X_t(\omega)$ is $\mathcal{B}([0,1]) \otimes \mathcal{F}$ - measurable.

a) Show that the map $t \mapsto X_t$ is continuous in $L^2(\mathbb{P})$ if and only if K is continuous in $[0,1]^2$. In the following we suppose that this condition is satisfied.

b) Let $h : [0,1] \to \mathbb{R}$ be a Borel-measurable function satisfying $\int_0^1 |h(t)|\sqrt{K(t,t)}dt < \infty$. Show that for almost all $\omega \in \Omega$ the integral $\int_0^1 h(t)X_t(\omega)dt$ is absolutely convergent.

c) Assume that h from b) also satisfies $\int_0^1 |h(t)|dt < \infty$. Show that $Z := \int_0^1 h(t)X_t dt$ is the $L^2(\mathbb{P})$-limit of the random variables

$$Z_n := \sum_{i=1}^{n} X_{\frac{i}{n}} \int_{\frac{i-1}{n}}^{\frac{i}{n}} h(t)dt \qquad \text{as } n \to \infty.$$

Deduce that Z is a centered Gaussian random variable and determine its variance.

d) Assume that K is twice continuously differentiable. Show that for all $t \in [0,1]$ the limit

$$\hat{X}_t := \lim_{s\to t} \frac{X_s - X_t}{s - t} \qquad \text{exists in } L^2(\mathbb{P}).$$

Show that $(\hat{X}_t)_{t\in[0,1]}$ is a centered Gaussian process and compute its covariance function.

Please hand in your solutions on Monday, 4 February, 2019 at 9:15 am.