Stochastik II
Wintersemester 2018/19
Humboldt-Universität zu Berlin
Prof. Dr. Dörte Kreher

Exercise Sheet 3

Problem 1 (4 Points)
Let X, Y be two i.i.d. random variables on $(\Omega, \mathcal{F}, \mathbb{P})$, being exponentially distributed with parameter 1. Find the regular conditional distribution of X given $Z := \min(X, Y)$.

Problem 2 (4 Points)
A box contains b black and w white balls ($b, w \geq 1$). The content of the box is changed as follows: One ball is randomly drawn, its color is denoted, and the ball is placed back into the box together with c other balls of the same color ($c \in \mathbb{N}_0$ fixed). This procedure is repeated infinitely often. We denote by X_n the number of black balls and by Y_n the number of white balls in the box after the n-th turn, $n \in \mathbb{N}_0$. Set $\mathcal{F}_n := \sigma(X_0, X_1, \ldots, X_n)$ for all $n \in \mathbb{N}_0$.

a) Show that the proportions

$$V_n := \frac{X_n}{X_n + Y_n}, \quad n \in \mathbb{N}_0,$$

of the black balls in the box form a martingale with respect to $(\mathcal{F}_n)_{n\in\mathbb{N}_0}$.

b) Can $(V_n)_{n\in\mathbb{N}_0}$ be represented as a running sum of mutually independent random variables?

c) Define τ to be the first time a black ball is drawn. Is $\tau < \infty$ almost surely?

Problem 3 (4 Points)
Let $(\Omega, \mathcal{F}, (\mathcal{F}_n)_{n\in\mathbb{N}_0}, \mathbb{P})$ be a filtered probability space.

a) Let (X_n) be a supermartingale with respect to $(\mathcal{F}_n)_{n\in\mathbb{N}_0}$ such that $\mathbb{E}(X_n) = \mathbb{E}(X_0)$ for all $n \in \mathbb{N}$. Show that $(X_n)_{n\in\mathbb{N}_0}$ is a martingale.

b) Let (X_n) be a martingale with respect to $(\mathcal{F}_n)_{n\in\mathbb{N}_0}$ such that $X_n \geq 0$ almost surely for all $n \in \mathbb{N}_0$. Prove that for $\mathbb{P}$-almost all $\omega \in \Omega$ it holds that

$$X_n(\omega) = 0 \text{ for some } n \quad \Rightarrow \quad X_{n+k}(\omega) = 0 \text{ for all } k \in \mathbb{N}_0.$$

Problem 4 (4 Points)
Let $(X_n)_{n\in\mathbb{N}_0}$ be a supermartingale on the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_n)_{n\in\mathbb{N}_0}, \mathbb{P})$. We assume that the random variables X_n and X_m have the same distribution for all $n, m \in \mathbb{N}_0$. Show that:

a) $(X_n)_{n\in\mathbb{N}_0}$ is a martingale.

b) For every $a \in \mathbb{R}$ the processes $(X_n \wedge a)_{n\in\mathbb{N}_0}$ and $(X_n \vee a)_{n\in\mathbb{N}_0}$ are also martingales.

c) For all $n > m$ and $a \in \mathbb{R}$, $X_n \geq a$ almost surely on the event $\{X_m \geq a\}$.

d) $X_n = X_0$ almost surely for all $n \in \mathbb{N}$.

Please hand in your solutions on Monday, 12 November, 2018 at 9:15 am.