Stochastik II
Wintersemester 2018/19
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Exercise Sheet 4

Problem 1 (3 Points)
Let τ be a random variable on $(\Omega, \mathcal{F}, \mathbb{P})$ with values in $\mathbb{N}_0$. We define

$$X_n := \mathbb{1}_{[0,n]}(\tau) \quad \text{and} \quad \mathcal{F}_n := \sigma(X_0, \dots, X_n), \quad n \in \mathbb{N}_0.$$

a) Show that $(X_n)_{n\in\mathbb{N}_0}$ is a submartingale with respect to $(\mathcal{F}_n)_{n\in\mathbb{N}_0}$.

b) We denote by $X_n = X_0 + M_n + A_n,\ n \in \mathbb{N}_0$, the Doob decomposition of X, where M is a martingale and A is a predictable process. Show that

$$A_n = \sum_{k=1}^{\tau\wedge n} \mathbb{P}(\tau = k | \tau > k-1), \quad n \in \mathbb{N}_0.$$

Problem 2 (2 Points)
Let τ be a stopping time on the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_n)_{n\in\mathbb{N}_0}, \mathbb{P})$. Prove that

$$\mathcal{F}_\tau = \left\{ \bigcup_{n\in\overline{\mathbb{N}}_0} A_n \cap \{\tau = n\} \,\middle|\, A_\infty \in \mathcal{F} \text{ and } A_n \in \mathcal{F}_n \ \forall\, n \in \mathbb{N}_0 \right\}.$$

Problem 3 (3 Points)
Let τ be a stopping time on the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_n)_{n\in\mathbb{N}_0}, \mathbb{P})$. Prove that if $(X_n)_{n\in\mathbb{N}_0}$ and $(Y_n)_{n\in\mathbb{N}_0}$ are two $(\mathcal{F}_n)$-martingales and $X_\tau = Y_\tau$ $\mathbb{P}$-a.s. on the set $\{\tau < \infty\}$, then the process

$$Z_n := X_n \mathbb{1}_{\{\tau > n\}} + Y_n \mathbb{1}_{\{\tau \leq n\}}, \quad n \in \mathbb{N}_0,$$

is again an $(\mathcal{F}_n)$-martingale.

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Problem 4 (8 Points)
Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of i.i.d. random variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ such that $\mathbb{P}(X_n = 1) = p \in (0,1)$ and $\mathbb{P}(X_n = -1) = 1 - p$. We define the filtration $\mathcal{F}_0 := \{\emptyset, \Omega\}$, $\mathcal{F}_n := \sigma(X_1, \dots X_n)$, $n \in \mathbb{N}$, and set $S_n := \sum_{k=1}^{n} X_k$ for all $n \in \mathbb{N}$ with $S_0 := 0$. For $c \in \mathbb{Z}$ we define the first hitting time $\tau_c := \inf\{n \in \mathbb{N}_0 : \ S_n = c\}$. Let $a, b \in \mathbb{N}$ and define $\tau := \tau_a \wedge \tau_{-b}$.

a) Prove that $\mathbb{P}(\tau < \infty) = 1$.

b) Suppose that $p = 1/2$.
   (i) Apply the stopping theorem to (S_n) and compute $\mathbb{P}(\tau_a < \tau_{-b})$ as a function of a, b.
   (ii) Prove that $M_n := S_n^2 - n$, $n \in \mathbb{N}_0$, is a martingale with respect to $(\mathcal{F}_n)_{n\in\mathbb{N}_0}$.
   (iii) Apply the stopping theorem to (M_n) and compute $\mathbb{E}(\tau)$ as a function of a, b.
   (iv) Show that $\mathbb{P}(\tau_a < \infty) = 1$ and $\mathbb{E}(\tau_a) = \infty$.

c) Suppose that $p \neq 1/2$.
   (i) Find $z > 0$, $z \neq 1$ such that $(z^{S_n})_{n\in\mathbb{N}_0}$ is a martingale with respect to $(\mathcal{F}_n)_{n\in\mathbb{N}_0}$.
   (ii) Compute $\mathbb{P}(\tau_a < \tau_{-b})$ and $\mathbb{P}(\tau_a > \tau_{-b})$ as functions of a, b.
   (iii) Show that for $p < 1/2$ we have

$$\mathbb{P}(\tau_a < \infty) = \left(\frac{p}{1-p}\right)^a \quad \text{and} \quad \mathbb{P}(\tau_{-b} < \infty) = 1,$$

   and for $p > 1/2$ we have

$$\mathbb{P}(\tau_a < \infty) = 1 \quad \text{and} \quad \mathbb{P}(\tau_{-b} < \infty) = \left(\frac{1-p}{p}\right)^b.$$