Stochastik II
Wintersemester 2018/19
Humboldt-Universität zu Berlin
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Exercise Sheet 5

Problem 1 (4 Points)
Let $Y_0 \in \mathbb{R}_+$ and let $(Y_n)_{n\in\mathbb{N}} \subset L^1(\Omega, \mathcal{F}, \mathbb{P})$ be i.i.d. random variables. For $0 < \beta < 1$ define

$$R_n := \sum_{k=0}^{n} \beta^{k-n} Y_k,\ n \in \mathbb{N}_0, \qquad X := \sum_{k=1}^{\infty} \beta^k Y_k.$$

a) Prove that $(R_n)_{n\in\mathbb{N}_0}$ solves the recursive equation $R_{n+1} = \beta^{-1} R_n + Y_{n+1},\ n \in \mathbb{N}_0$, with initial value $R_0 = Y_0$, and show that $\lim_n \beta^n R_n = Y_0 + X$.

b) Let F be the distribution function of X and let $u(x) := F(-x),\ x \in \mathbb{R}$. Show that $(u(R_n))_{n\in\mathbb{N}_0}$ is a martingale with respect to $\mathcal{F}_n := \sigma(Y_0, \ldots, Y_n),\ n \in \mathbb{N}_0$, which is closed by $Z := \mathbb{1}_{\{X \leq -Y_0\}}$.

c) Show that for $\tau := \min\{n \geq 0 :\ R_n \leq 0\}$ we have $\mathbb{P}(\tau < \infty) \leq \frac{u(Y_0)}{u(0)}$.

Problem 2 (4 Points)
On a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_n)_{n\in\mathbb{N}_0}, \mathbb{P})$ let $(X_n)_{n\in\mathbb{N}_0} \subset L^1$ and $(Y_n)_{n\in\mathbb{N}_0}$ be two sequences of positive, $(\mathcal{F}_n)$-adapted random variables. Suppose that

$$\sum_{n=0}^{\infty} Y_n < \infty \quad \text{and} \quad \mathbb{E}(X_{n+1}|\mathcal{F}_n) \leq (1 + Y_n) X_n \text{ a.s.} \quad \forall\, n \in \mathbb{N}_0.$$

Show that $(X_n)_{n\in\mathbb{N}_0}$ converges almost surely.

Problem 3 (4 Points)
Let $(X_n)_{n\in\mathbb{N}_0}$ be a sequence of random variables with values in $[0, 1]$. We set $\mathcal{F}_n := \sigma(X_0, \ldots, X_n)$, $n \in \mathbb{N}_0$. Assume that $X_0 = a \in [0, 1]$ and

$$\mathbb{P}\left(X_{n+1} = \frac{X_n}{2} \middle| \mathcal{F}_n \right) = 1 - X_n, \quad \mathbb{P}\left(X_{n+1} = \frac{1 + X_n}{2} \middle| \mathcal{F}_n \right) = X_n, \quad n \in \mathbb{N}_0.$$

a) Show that $(X_n)_{n\in\mathbb{N}_0}$ is a martingale converging almost surely and in L^2 to a random variable X_∞.

b) Show that

$$\mathbb{E}\left[(X_{n+1} - X_n)^2\right] = \frac{1}{4} \mathbb{E}\left[X_n(1 - X_n)\right],\ n \in \mathbb{N}_0.$$

c) Compute $\mathbb{E}[X_\infty(1 - X_\infty)]$. What is the law of X_∞?

Problem 4 (4 Points)

Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of independent random variables on $(\Omega, \mathcal{F}, \mathbb{P})$ with

$$\mathbb{P}(X_n = 1) = \mathbb{P}(X_n = -1) = \frac{1}{2n}, \qquad \mathbb{P}(X_n = 0) = 1 - \frac{1}{n}, \quad n \in \mathbb{N}.$$

We define a process $(Y_n)_{n\in\mathbb{N}}$ recursively via $Y_1 := X_1$ and for $n \geq 2$,

$$Y_n := X_n \mathbb{1}_{\{Y_{n-1}=0\}} + nY_{n-1}|X_n|\mathbb{1}_{\{Y_{n-1}\neq 0\}}.$$

Show that:

a) $(Y_n)_{n\in\mathbb{N}}$ is a martingale with respect to its natural filtration.

b) $(Y_n)_{n\in\mathbb{N}}$ converges in probability.

c) $(Y_n)_{n\in\mathbb{N}}$ does not converge almost surely.

Please hand in your solutions on Monday, 26 November, 2018 at 9:15 am.