Stochastik II
Wintersemester 2018/19
Humboldt-Universität zu Berlin
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Exercise Sheet 6

Problem 1 (4 Points)
Show that a martingale $(X_n)_{n\in\mathbb{N}}$ is uniformly integrable if and only if the family

$$\{X_\tau \mid \tau \text{ is a bounded stopping time}\}$$

is uniformly integrable.

Problem 2 (4 Points)
Let $\Omega = [0,1]$, $\mathcal{F} = \mathcal{B}([0,1])$, $\mathbb{P}$ the Lebesgue measure on $[0,1]$ and $f : [0,1] \mapsto \mathbb{R}$ be a Lipschitz continuous function. Define for all $n \in \mathbb{N}$,

$$Y_n := \sum_{k=1}^{2^n} \frac{k-1}{2^n} \mathbf{1}_{A_{k,n}},$$

$\mathcal{F}_n := \sigma(Y_1, \ldots, Y_n)$ and $X_n = 2^n(f(Y_n + 2^{-n}) - f(Y_n))$, where $A_{k,n} = \left[\frac{k-1}{2^n}, \frac{k}{2^n}\right)$ for $1 \le k \le 2^n$.

a) Show that $(X_n)_{n\in\mathbb{N}}$ is a uniformly integrable martingale with respect to $(\mathcal{F}_n)_{n\in\mathbb{N}}$ that converges $\mathbb{P}$-a.s and in L^1.

b) Conclude that there exists a Lebesgue integrable function g such that

$$f(x) - f(0) = \int_0^x g(y)\, dy \quad \forall\ 0 \le x \le 1.$$

Problem 3 (4 Points)
Let $\{(X_n, Y_n) : n \in \mathbb{N}_0\}$ be a sequence of $\mathbb{Z}^2$-valued i.i.d. random vectors with $\mathbb{E}(X_1) = \mathbb{E}(Y_1) = 0$, $0 < \mathbb{E}(X_1^4), \mathbb{E}(Y_1^4) < \infty$, and $\mathbb{E}(X_1 Y_1) = c$. For $n \in \mathbb{N}$ we define

$$(U_{n+1}, V_{n+1}) = (U_n, V_n) + (X_{n+1}, Y_{n+1}) \quad \text{with} \quad (U_0, V_0) \in \mathbb{N}^2.$$

Let $\tau := \min\{n \ge 0 : \ U_n V_n = 0\}$ be the first time the two dimensional random walk (U_n, V_n) hits one (or both) of the two axes in $\mathbb{R}^2$. Show that:

a) The process $M_n := U_n V_n - cn,\ n \in \mathbb{N}_0$, is a martingale with respect to the natural filtration of the process $(X_n, Y_n),\ n \in \mathbb{N}_0$.

b) $\tau < \infty$ almost surely.

c) $\mathbb{E}(\tau) < \infty$ if and only if $c < 0$, and in this case $\mathbb{E}(\tau) = -\frac{U_0 V_0}{c}$.

Problem 4 (4 Points)
Let $(X_n)_{n\in\mathbb{N}_0}$ be a sequence of random variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, converging almost surely to X. Let $Y \in L^1$ with $|X_n| \le Y$ a.s. for all $n \in \mathbb{N}_0$. Moreover, let $(\mathcal{F}_n)_{n\in\mathbb{N}_0}$ be an arbitrary filtration of $\mathcal{F}$ and set $\mathcal{F}_\infty := \sigma(\cup_{k=0}^{\infty}\mathcal{F}_k)$. For any $n \in \mathbb{N}_0$ we define $Y_n := \sup_{k,l\ge n} |X_k - X_l|$ and $Z_n := \mathbb{E}[Y_n|\mathcal{F}_n]$.

a) Show that the process $(Z_n)_{n\in\mathbb{N}_0}$ is a uniformly integrable supermartingale and investigate its convergence.

b) Show that $|\mathbb{E}(X_n|\mathcal{F}_n) - \mathbb{E}(X|\mathcal{F}_n)| \to 0$ almost surely as $n \to \infty$ and decude that

$$\mathbb{E}[X_n|\mathcal{F}_n] \longrightarrow \mathbb{E}[X|\mathcal{F}_\infty] \quad \text{a.s. for } n \to \infty.$$

Please hand in your solutions on Monday, 3 December, 2018 at 9:15 am.