Stochastik II
Wintersemester 2018/19
Humboldt-Universität zu Berlin
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Exercise Sheet 7

Problem 1 (4 Points)
Let $(Y_n)_{n\in\mathbb{N}}$ be a sequence of positive, independent random variables with $\mathbb{E}(Y_n) = 1$ for all $n \in \mathbb{N}$. Let $\mathcal{F}_0 := \{\emptyset, \Omega\}$ and $\mathcal{F}_n := \sigma(Y_1, \dots, Y_n),\ n \in \mathbb{N}$. We consider the product martingale $X_0 := 1$ and $X_n := \prod_{k=1}^n Y_k,\ n \in \mathbb{N}$.

a) Show that $(\sqrt{X_n})_{n\in\mathbb{N}_0}$ is a supermartingale.

b) Assume that $\prod_{k=1}^\infty \mathbb{E}(\sqrt{Y_k}) = 0$. Investigate the convergence and the limit of $(\sqrt{X_n})_{n\in\mathbb{N}_0}$ and then of $(X_n)_{n\in\mathbb{N}_0}$. Is the martingale $(X_n)_{n\in\mathbb{N}_0}$ uniformly integrable?

c) Assume that $\prod_{k=1}^\infty \mathbb{E}(\sqrt{Y_k}) > 0$. Show that $(\sqrt{X_n})_{n\in\mathbb{N}_0}$ is a Cauchy sequence in L^2 and deduce that $(X_n)_{n\in\mathbb{N}_0}$ is uniformly integrable.

Problem 2 (4 Points)
Let $(\Omega, \mathcal{F})$ be a measurable space and $\mathbb{P}$, $\mathbb{Q}$ two probability measures on $\mathcal{F}$. We define the relative entropy of $\mathbb{Q}$ with respect to $\mathbb{P}$ by

$$H(\mathbb{Q}|\mathbb{P}) := \begin{cases} \mathbb{E}_{\mathbb{Q}}[\ln \frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}] & \text{if } \mathbb{Q} \ll \mathbb{P} \\ +\infty & \text{otherwise.} \end{cases}$$

a) Show that $H(\mathbb{Q}|\mathbb{P})$ is well defined and always non-negative.

b) Show that $H(\mathbb{Q}|\mathbb{P}) = 0$ if and only if $\mathbb{Q} = \mathbb{P}$.

c) Is $H(\cdot|\cdot)$ a metric on the set of all probability measures on $\mathcal{F}$?

Problem 3 (4 Points)
Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $\mathbb{Q}_1$ and $\mathbb{Q}_2$ be measures on $(\Omega, \mathcal{F})$ such that $\mathbb{Q}_1 \ll \mathbb{P}$ and $\mathbb{Q}_2 \ll \mathbb{P}$. Show that

$$\sup_{A\in\mathcal{F}} |\mathbb{Q}_1(A) - \mathbb{Q}_2(A)| = \frac{1}{2} \cdot \mathbb{E}\left[\left|\frac{d\mathbb{Q}_1}{d\mathbb{P}} - \frac{d\mathbb{Q}_2}{d\mathbb{P}}\right|\right].$$

Problem 4 (4 Points)
Henry plays the following betting game: At the beginning of every round k he wagers a certain amount Z_k. Then a fair coin is flipped. If it shows head, Henry gets Z_k. If it show tails, he loses Z_k. Henry has developed the following strategy for playing the game: At the beginning he chooses a sequence of n positive numbers $x_1, \dots, x_n$. In the first round he wagers the sum of the first and last numbers, i.e. $Z_1 = x_1 + x_n$. If he wins, he deletes those tow numbers; if he loses he appends their sum as an extra term $x_{n+1} := x_1 + x_n$ at the right end of the sequence. He plays iteratively according to this set of rules. If at some point the sequence only contains one number, he wagers that amount in the next round. In this case, if he wins, he deletes this number, and if he loses, he appends the number to the sequence to obtain two terms again. If all numbers are deleted, he stops playing. Show that:

a) The game terminates almost surely in finite time with a profit of $\sum_{i=1}^{n} x_i$ for Henry.

b) The time until termination has finite mean.

c) The expected size of his largest cumulated stake before winning is infinite.

Please hand in your solutions on Monday, 10 December, 2018 at 9:15 am.