Stochastik II
Wintersemester 2018/19
Humboldt-Universität zu Berlin
Prof. Dr. Dörte Kreher

Exercise Sheet 8

Problem 1 (4 Points)
Let $(X_k)_{k\in\mathbb{N}}$ be i.i.d. random variables with $\mathbb{P}(X_k = 1) = p = 1 - \mathbb{P}(X_k = -1)$ for some $p \in (0,1)$. Consider the simple random walk $S_n := \sum_{k=1}^n X_k,\ n \in \mathbb{N}$, starting from $S_0 = 0$. For each $n \in \mathbb{N}_0$ we define $M_n := \max\{S_k : 0 \le k \le n\}$, $Y_n := M_n - S_n$, and $A_n := |S_n|$.

a) Show that $Y = (Y_n)_{n\in\mathbb{N}_0}$ is an $(\mathcal{F}_n^S)$-Markov chain and find its transition probabilities.

b) Show that $A = (A_n)_{n\in\mathbb{N}_0}$ is not an $(\mathcal{F}_n^S)$-Markov chain for $p \neq \frac{1}{2}$, but it is a Markov chain in its own filtration $(\mathcal{F}_n^Y)$ and find its transition probabilities.

Problem 2 (4 Points)
An urn initially contains b black balls and w white balls. A ball is picked at random. If it is black, it is removed together with a white ball. If it is white, then it is replaced into the urn together with another white and a black ball. This is repeated until there are no black or no white balls left in the urn.

a) Compute the probability that the process terminates in finite time, if $b = n$ and $w = n+2$ for some $n \in \mathbb{N}$.

b) Show that the process will terminate almost surely, if $b = n+2$ and $w = n$ for some $n \in \mathbb{N}$ and compute the expected number of iterations until termination.

Problem 3 (4 Points)
Let $X = (X_n)_{n\in\mathbb{N}_0}$ be a Markov chain with state space $\mathbb{N}_0$ and transition probabilities

$$p_{01} = 1, \quad p_{i,i-1} + p_{i,i+1} = 1, \quad p_{i,i+1} = \left(\frac{i+1}{i}\right)^{\alpha} p_{i,i-1}, \quad i \ge 1,$$

for some $\alpha \in \mathbb{R}_+$. Suppose that $X_0 = 0$ almost surely.

a) Find the probability that X will never return to zero.

b) Find the probability that $X_n \to \infty$ as $n \to \infty$.

Problem 4 (4 Points)
Let X be a discrete time Markov chain, taking values in a countable state space E. Define the first hitting time of state i by $T_i := \inf\{n \ge 1 :\ X_n = i\}$ with $\inf \emptyset := \infty$. A pair of two distinct states $i, j \in E$ is called symmetric if

$$\mathbb{P}(T_j < T_i | X_0 = i) = \mathbb{P}(T_i < T_j | X_0 = j).$$

Show that if $X_0 = i$ and i, j are symmetric, the expected number of visits to j before the chain revisits i is 1.

Please hand in your solutions on Monday, 17 December, 2018 at 9:15 am.