Problem Set 1

Differential Geometry
WS 2019/20

Problem 1
(a) Let \( \tilde{\gamma} : [a, b] \to \mathbb{R}^n \) be a regular, differentiable curve and denote by \( \gamma : [0, \ell(\gamma)] \to \mathbb{R}^n \) its arclength reparametrization. Show that \( \gamma \) is differentiable and \( \|\dot{\gamma}(t)\| = 1 \) for all \( t \in [0, \ell(\gamma)] \).
(b) What can be said if \( \tilde{\gamma} \) is not necessarily regular but nowhere constant?

Problem 2
(a) Let \( \tilde{\gamma} : [a, b] \to \mathbb{R}^2 \) be a regular, twice differentiable curve. Compute the curvature \( \kappa : [a, b] \to \mathbb{R} \) in terms of its first and second derivative.
(b) Let \( f : [a, b] \to \mathbb{R} \) be a \( C^2 \)-function; let \( \gamma : [a, b] \to \mathbb{R}^2 \) be given by \( \gamma(t) = (t, f(t)) \). Derive formulas for the length of \( \gamma \) and its curvature. Show that the curvature is negative, positive, zero exactly where \( f \) is concave, convex or has an inflection point, respectively.
(c) Compute the turning number of \( \gamma \) in (b).

Problem 3
(a) Compute the length and the curvature of the following curves:
\( \alpha : [a, b] \to \mathbb{R}^3; \alpha(t) = (r \cos t, r \sin t, kt) \) (the helix)
\( \beta : (0, \pi) \to \mathbb{R}^2; \beta(t) = (\sin t, \cos t + \ln \tan(t/2)) \) (tractrix).
\( \gamma : (-1, 1) \to \mathbb{R}^2; \gamma(t) = (t^2, t^3) \) (semicubic parabola).
\( \delta : [0, 2\pi] \to \mathbb{R}^2; \delta(t) = (\cos t, \sin(2t)) \) (lemniscate of Gerono).
(b) Compute the turning number of \( \delta \).