Problem Set 4
Differential Geometry
WS 2019/20

Problems 1 and 2 can be discussed in class.
You may submit solutions for Problems 3 and 4 until November 20.

**Problem 1**
(a) Let \( \gamma : [a,b] \to \mathbb{R}^3 \) be a regular parametrized curve (not necessarily arc-length parametrized) with nowhere vanishing curvature. Show that \( \dddot{\gamma} \) is perpendicular to the binormal.
(b) Derive formulas for curvature, torsion and the reference frame for \( \gamma \) as in (a).
(c) Compute curvature, torsion and the reference frame for the helix.
(d) Show Frenet’s formulas.

**Problem 2** [Spherical geometry]
(a) Show that for \( P,Q \in S^2 \) the shorter arc of the circle with center 0 and radius 1 is the shortest regular curve on \( S^2 \) connecting \( P \) and \( Q \).
(b) For \( P,Q \in S^2 \) we define \( d_{S^2}(P,Q) = \angle(POQ) \).
Show that \( d_{S^2} \) defines a metric on \( S^2 \).
(b) For a curve \( \gamma : [a,b] \to S^2 \) show that
\[
\ell(\gamma) = \sup \left\{ \sum_{i=1}^{m} d_{S^2}(\gamma(t_i), \gamma(t_{i-1})) \mid a = t_0 < t_1 < \ldots < t_m = b \right\}.
\]
where \( \ell(\gamma) \) is the length of \( \gamma \) considered as a space curve.

**Problem 3**
Show that the total angle of a closed polygon is equal to \( 2\pi \) if and only if the polygon is planar, simple and convex.

**Problem 4**
Denote by \( \mathbb{H} : \{ (x_1, x_2) \mid x_2 > 0 \} \subset \mathbb{R}^2 \) the (open) upper half plane. The hyperbolic length \( \ell_{\mathbb{H}}(\gamma) \) of a \( C^1 \)-curve \( \gamma : [a,b] \to \mathbb{H} \) is defined to be
\[
\ell_{\mathbb{H}}(\gamma) = \int_{a}^{b} \frac{1}{x_2(t)} \| \dot{\gamma}(t) \| dt
\]
where \( \gamma(t) = (x_1(t), x_2(t)) \) are the components and \( \| . \| \) denotes the euclidean norm of \( \mathbb{R}^2 \).
(a) Explain that \( \ell \) does not change under reparametrization of \( \gamma \).
(b) Let \( P = (x_1, x_2), Q = (y_1, y_2) \in \mathbb{H} \) be two points in \( \mathbb{H} \). Show that the shortest \( C^1 \)-curve in \( \mathbb{H} \) joining \( P \) and \( Q \) is given by the euclidean segment \( PQ \) if \( x_1 = y_1 \) or the segment between \( P \) and \( Q \) of the unique euclidean circle containing \( P \) and \( Q \) whose center lies on the \( x_1 \)-axis if \( x_1 \neq y_1 \).
(c) Define \( d_{\mathbb{H}}(P,Q) \) to be the shortest hyperbolic length of a \( C^1 \)-curve in \( \mathbb{H} \) connecting \( P,Q \in \mathbb{H} \). Show that this defines a metric on \( \mathbb{H} \). Show that it is complete.
(d) Show that for a \( C^1 \)-curve \( \gamma : [a,b] \to \mathbb{H} \)
\[
\ell_{\mathbb{H}}(\gamma) = \sup \left\{ \sum_{i=1}^{m} d_{\mathbb{H}}(\gamma(t_i), \gamma(t_{i-1})) \mid a = t_0 < t_1 < \ldots < t_m = b \right\}.
\]
(e) What should the measure of an angle in \( \mathbb{H} \) analogous to a euclidean angle be? What are convex sets and therefore convex polygons of hyperbolic segments? Explain that the total angle of a simple closed convex hyperbolic polygon is bigger than \( 2\pi \) on one example. Can you prove it to be true in general?