Problems 1 to 4 can be discussed in the tutorial.
You may submit solutions for Problems 5 and 6 until January 8.

**Problem 1**
Prove the following statement: Let $F \subset \mathbb{R}^3$ be a regular oriented surface, $N : F \to S^2$ be its Gauss map. Let $\gamma : I \to F$ be an arc-length parametrized curve. Then $\gamma$ is a line of curvature if and if there is a function $\lambda : I \to \mathbb{R}$ satisfying
\[
\frac{d}{dt} N(\gamma(t)) = \lambda(t) \dot{\gamma}(t)
\]
for all $t \in I$.

**Problem 2**
Prove the following formula of Euler: Let $X_1, X_2 \in T_pF$ be eigenvectors of the Weingarten map of a regular surface $F \subset \mathbb{R}^3$, $p \in F$, $\|X_i\| = 1$ for $i = 1, 2$ and $\langle X_1, X_2 \rangle = 0$ (Why is this possible?). Denote by $\kappa_1, \kappa_2$ be the corresponding eigenvalues. Consider a vector of unit length $X \in T_pF$. Then $X := \lambda_1 X_1 + \lambda_2 X_2$ with $\lambda_1^2 + \lambda_2^2 = 1$. Show that
\[
II_p(X, X) = \lambda_1^2 \kappa_1 + \lambda_2^2 \kappa_2.
\]
Conclude that the principal curvatures are the extremal values of $\{II_p(X, X) \mid X \in T_pF, \|X\| = 1\}$.

**Problem 3**
Prove Lemma 52: Any regular surface $F \subset \mathbb{R}^3$ can be locally modelled around a point as the graph of a differentiable function over the tangent space at this point: If $N(p)$ is a normal at $p$, then there exist open neighbourhoods $U \subset T_pF \cong \mathbb{R}^2$ of 0, $V \subset \mathbb{R}^3$ of $p$ and a differentiable map $h : U \to \mathbb{R}$, such that $F \cap V = \{p + x + h(x)N(p) \mid x \in U\}$.

Hint: In the lemma we stated that the orthogonal projection to $T_pF$ restricted to $V \cap F$ is a diffeomorphism onto its image form which the claim follows (how?). Als note that in this formulation $T_pF$ is considered to be a vector subspace of $\mathbb{R}^3$ rather than an affine subspace parallel to it passing through $p$. Hence $p$ had to be added.

This problem may not be discussed in the tutorial since it is a result from Analysis II (implicit function theorem etc.) You are expected to know how to solve it, so if in doubt, please ask.

**Problem 4**
Let $F \subset \mathbb{R}^3$ be a regular surface, $p \in F$. Show:
(i) If $K(p) > 0$ or a sufficiently small neighbourhood $V \subset \mathbb{R}^3$ of $p$, $F \cap V$ lies on one side of the affine tangent plane $p + T_pF$ (a side of the hyperplane is given by $\{x \in \mathbb{R}^3 \mid (x - p, n) \geq 0\}$
for a normal $n \neq 0$ to $F$ at $p$.
(ii) If $K(p) < 0$ than for any neighbourhood $V \subset \mathbb{R}^3$ of $p$ $F \cap V$ lies on both sides of the affine tangent plane $p + T_pF$.
(iii) Explain why either of the two statement is false in the case $K(p) = 0$. 
Problem 5
Let \( f : U \subset \mathbb{R}^3 \to \mathbb{R} \) be a differentiable function (at least \( C^2 \)) for which \( 0 \in \mathbb{R} \) is a regular value, i.e. for all \( x \in f^{-1}(0) \), \( d_x f \neq 0 \). Explain why \( F := f^{-1}(0) \subset \mathbb{R}^3 \) is a regular surface. Compute second fundamental form and Weingarten map in terms of \( f \) and its derivatives. Derive formulas for its Gaussian and its mean curvature.

Problem 6
Let \( F \subset \mathbb{R}^3 \) be a compact surface. Prove the following statements:
(i) The Gauss map is surjective.
(ii) If the Gauss map is injective then \( K \geq 0 \) everywhere.
(iii) Show that the restriction of the Gauss map to the set \( \{ p \in F \mid K(p) \geq 0 \} \) is surjective.
(iv) Show that the restriction of the Gauss map to the set \( \{ p \in F \mid K(p) > 0 \} \) is a local diffeomorphism. Hint: Problem 5 of Problem Set 8 could be helpful.