Problem Set 1

Differential Geometry II
Summer 2020

Problem 1
Prove Proposition 5 of the lecture, in particular (3):
We have the following relations

$$\begin{aligned}
&\text{for } v, w \in V, \alpha \in \Lambda^k(V^*): \\
&\qquad v \lrcorner (w \lrcorner \alpha) = -w \lrcorner (v \lrcorner \alpha) \\
&\text{for } v \in V, \alpha \in \Lambda^k(V^*),\ \beta \in \Lambda^\ell(V^*): \\
&\qquad v \lrcorner (\alpha \wedge \beta) = (v \lrcorner \alpha) \wedge \beta + (-1)^k \alpha \wedge (v \lrcorner \beta) \\
&\text{for } F : V \to W \ \text{ linear, } v \in V, \alpha \in \Lambda^k(V^*): \\
&\qquad v \lrcorner (F^* \alpha) = F^*(F(v) \lrcorner \alpha).
\end{aligned}$$

Problem 2
Prove Lemma 6 of the lecture:
(1) The definition of dV is independent of the choice of an oriented orthonormal basis.
(2) It has length one: $\langle dV, dV \rangle = 1$ and $\Lambda^n(V) = \mathbb{R} dV$.
(3) For the dual basis $\{\alpha_1, ..., \alpha_n\}$ of a oriented orthonormal basis as above we have

$$dV = \alpha_1 \wedge ... \wedge \alpha_n.$$

Problem 3
Prove Lemma 7 of the lecture:
(1) The map

$$* : \Lambda^k(V^*) \longrightarrow \Lambda^{n-k}(V^*)$$

is an isometry which is referred to as **Hodge-$*$-operator**.
(2) On k–forms $*^2 = * \circ * = (-1)^{k(n-k)}$.
(3) For $\alpha, \beta \in \Lambda^k(V^*)$ we have

$$\alpha \wedge *\beta = \langle \alpha, \beta \rangle dV.$$

Problem 4
Study problems 1,2,3 and 5 of the book of Agricola and Friedrich, pages 8 and 9

Problem 5
Prove Cartan's Lemma (Problem 4 of the book of Agricola and Friedrich, page 9)

Problem 6
Study problem 8 of the book of Agricola and Friedrich, page 9 for the case of a euclidean 4–dimensional vector space V, i.e. $q = 0$ in their notation.

Problem 7
Study problems 6 and 7 of the book of Agricola and Friedrich, page 9.