Problem Set 10

Differential Geometry II
Summer 2020

Problem 1 [Quaternionic Hopf Bundle]
Let
$$S^7 := \{A \in M(2;\mathbb{C}) \mid \mathrm{Trace}(\overline{A}^T A) = 2\} \subset M(2;\mathbb{C}) \cong \mathbb{C}^4.$$
The Lie group $SU(2) := \{g \in M(2;\mathbb{C}) | \overline{g}^T g = \mathbf{E}_2, \det g = 1\}$ acts on it (from the right) via $A \mapsto Ag$.
(a) Construct a homeomorphism
$$S^7/SU(2) \cong S^4$$
such that the quotient map $S^7 \xrightarrow{\pi} S^4$ is smooth.
(b) Show that $S^7 \xrightarrow{\pi} S^4$ is a principal $SU(2)$–bundle.
(c) Verify that the orthogonal complements of the fibre tangents, $T_p^h S^7 := (T_p\pi^{-1}([p]))^\perp$, define a connection of the principal $SU(2)$–bundle.
(d) Compute the curvature of this connection. Determine the Chern classes of the Quaternionic Hopf bundle, i.e. the Chern classes of the associated complex vector bundle of rank 2 w.r.t. the representation $SU(2) \to Gl(2;\mathbb{C})$.

Problem 2 [Stiefel-Whitney Class]
(1) Show there exist exactly two real bundles of rank 3 up to isomorphism on S^2. If you can, generalize this to closed surfaces.
(2) Show that all real bundles over the 3-sphere are trivial.

Problem 3 [Gauge Theory]
Let $P \xrightarrow{\pi} M$ be a principal G-bundle over a manifold M for a Lie group G.
(a) Consider the quotient space $P \times_\alpha G := P \times G/\sim$ where the equivalence relation is given by
$$(p, h) \sim (pg, g^{-1}hg) \quad \forall g \in G.$$
Show that this is a fibre bundle so that there is a Lie group structure on each fibre which varies smoothly with the base point. The Lie algebras of these Lie groups are given by the associated Lie algebra bundle.
(b) Let $\mathcal{G}(P)$ denote the space of sections of $P \times_\alpha G$ which are called **gauge transformations** or the **gauge group** of P. Describe the elements as a subset of smooth maps $g : P \to G$. Using this identification show that for a connection $A \in \mathcal{C}(P)$ and a gauge transformation $g \in \mathcal{G}(P)$ by
$$A \cdot g := g^{-1}Ag + g^{-1}dg$$
where $g^{-1}Ag$ denotes the adjoint action $Ad_{g^{-1}}$ on the Lie algebra $\underline{g}$, we obtain a new connection. The sign of the second term is subtle. It should become clear by your calculations checking the porperties of a connection. Check that
$$A \cdot (gh) = (A \cdot g) \cdot h.$$
(c) Show that $D_{A\cdot g}(g^{-1}\alpha g) = g^{-1}(D_A\alpha)g$ gor any $\alpha \in \Omega^k(M; \underline{\mathbf{g}})$. Moreover, show
$$F_{A\cdot g} = g^{-1}F_A g.$$
Conclude that the Yang-Mills equation is invariant under the action of the gauge group.

Problem 4 [Minimal Surfaces]
There was a little confusion in the class on June 25:
What is actually needed in Step (i) of the proof of the formula for the first variation of the area functional on immersed surfaces in $\mathbb{R}^3$ (Proposition 82) is the following: Given a vector field X along u_0, i.e. asmooth function $X : F \to \mathbb{R}^3$ which is tangent to the fixed 1-manifold C of $\mathbb{R}^3$ at the boundary, there is a smooth family $u_\tau : F \to \mathbb{R}^3$ with $u_\tau|_{\partial F} : \partial F \to C$ diffeomorphisms such that

$$\frac{d}{d\tau}\Big|_{\tau=0} u_\tau = X.$$

Prove this statement. First explain that this it is satisfied if $X_p \perp d_p u_0(T_p F)$ everywhere. Next, if $X_p \in d_p u_0(T_p F)$ consider the flow of the uniquely determined vector field ξ on F with $X_p = d_p u_0(\xi_p)$. Finally, split an arbitrary vector field and combine both observations.

Problem 5 [Euler-Lagrange equations]
Let $L : TM \times \mathbb{R} \to \mathbb{R}$ be a smooth function on the tangent bundle TM of an n–manifold.
(1) Derive the coordinate form of the Euler-Lagrange equations for critical points of+ the Lagrange funtional

$$\frac{\partial L}{\partial x_j}(\gamma(t), \dot{\gamma}(t)) - \frac{d}{dt}\Big(\frac{\partial L}{\partial \dot{x}_j}(\gamma(t), \dot{\gamma}(t))\Big) = 0.$$

for all $j = 1, ..., n$.
(2) Show that the field of covectors of M along γ

$$\sum_{j=1}^{n} \Big(\frac{\partial L}{\partial x_j}(\gamma(t), \dot{\gamma}(t)) - \frac{d}{dt}\big(\frac{\partial L}{\partial \dot{x}_j}(\gamma(t), \dot{\gamma}(t))\big)\Big) d_{\gamma(t)} x^j$$

is independent of the chosen coordinates.