Problem Set 11

Differential Geometry II
Summer 2020

Problem 1 [Euler-Lagrange equations]
Let $L : TM \times \mathbb{R} \to \mathbb{R}$ be a smooth function on the tangent bundle TM of an n–manifold.
(1) Derive the coordinate form of the Euler-Lagrange equations for critical points of+ the Lagrange funtional

$$\frac{\partial L}{\partial x_j}(\gamma(t), \dot\gamma(t)) - \frac{d}{dt}\Big(\frac{\partial L}{\partial \dot x_j}(\gamma(t), \dot\gamma(t))\Big) = 0.$$

for all $j = 1, ..., n$.
(2) Show that the field of covectors of M along γ

$$\sum_{j=1}^{n}\Big(\frac{\partial L}{\partial x_j}(\gamma(t), \dot\gamma(t)) - \frac{d}{dt}\big(\frac{\partial L}{\partial \dot x_j}(\gamma(t), \dot\gamma(t))\big)\Big) d_{\gamma(t)} x^j$$

is independent of the chosen coordinates.
(3) Check the correctness of the form of the equation from class: is torsion freeness relavant? Express the equation

$$d_{(\gamma(t),\dot\gamma(t))} L_t \circ (d_{(\gamma(t),\dot\gamma(t)} \pi)^{-1} - \nabla^{\gamma}_{\frac{d}{dt}}(d^{v}_{(\gamma(t),\dot\gamma(t))} L_t) = 0$$

in local coordinates

Problem 2 [Jacobi Fields]
Let $\gamma : I \to M$ be a geodesic of the Riemannian manifold (M, g), ξ a Jacobi field along γ.
(1) Show that

$$\frac{d^2}{dt^2}\langle \xi, \dot\gamma\rangle = 0.$$

Conclude that
(a) $\xi(t) = \xi_0(t) + (at + b)\dot\gamma(t)$ with $a, b \in \mathbb{R}$ and a Jacobi field $\xi_0 \perp \dot\gamma$.
(b) If $\xi(t_0) = \xi(t_1)$ at different $t_0, t_1 \in I$, then $\xi, \nabla_{\dot\gamma}\xi \perp \dot\gamma$ everywhere.
(2) Show that the differential of the exponential map $d_X \exp_p(Y) = \eta(1)$ for a Jacobi field η along the geodesic $\gamma_X : t \in [0, 1] \mapsto \exp_p(tX)$ with $\eta(0) = 0$ and $\nabla_X \eta = Y$ for any $X \in T_pM$.

Problem 3 [Index Form]
Let $\gamma : [a, b] \to M$ be a geodesic of the Riemannian manifold (M, g). For continuous, piecewise smooth vector fields X, Y along γ we define the bilinear functional

$$I(X, Y) = \sum_{i=1}^{k}\langle \Delta_{t_i} X, Y\rangle + \int_a^b (\langle \nabla_{\dot\gamma} X, \nabla_{\dot\gamma} Y\rangle + \langle R(\dot\gamma, X)\dot\gamma, Y\rangle) dt.$$

where X is smooth on $[t_i, t_{i+1}]$ with $a = t_0 < t_1 < ...t_k < t_{k+1} = b$ and

$$\Delta_{t_i} X = \lim_{t \downarrow t_i} \nabla_{\dot\gamma} X(t) - \lim_{t \uparrow t_i} \nabla_{\dot\gamma} X(t) \in T_{\gamma(t_i)}M.$$

Show that X is a (smooth) Jacobi field if and only if $I(X, Y) = 0$ for all continuous, piecewise smooth vector fields Y along γ which vanish at the end points $t = a$ and $t = b$.

Problem 4 [Local Isometries]
The following two problems are related:
(1) Let $(M,g),(N,h)$ be Riemannian manifolds, N connected and $\varphi : (M,g) \to (N,h)$ a local isometry, i.e. $\varphi^* h = g$ everywhere. Prove: If M is complete, then φ is a covering.
(2) Show that if a connected, complete Riemannian manifold (M,g) has no conjugated points then its exponential map $\exp_p : T_pM \to M$ is a covering.
See Lemma 1.32 and Theorem 1.33. of Cheeger/Ebin: "Comparison Theorems in Riemannian Geometry", Elsevier 1975 (electronically available in our Library)