Problem Set 12

Differential Geometry II
Summer 2020

Problem 1 [Kähler Form]
(a) Let $(z_k = x_k + \mathrm{i}y_k)$ for $k = 1, ..., n$ be the complex coordinates of $\mathbb{C}^n$. Show that

$$\omega = \frac{\mathrm{i}}{2}\sum_{k=1}^{n} dz^k \wedge d\overline{z^k}$$

corresponds to the standard symplectic form of $\mathbb{R}^{2n}$ under an appropriate identification.
(b) Let (M, J, g) be a Kähler manifold. Show that its Kähler defines a symplectic form on M.

Problem 2 [Fubini Study Form]
Let $A \in \Omega^1(S^{2n+1}; \mathrm{i}\mathbb{R})$ be the connection one form on the total space of the Hopf bundle $S^{2n+1} \to \mathbb{C}P^n$ where the connection is given by the horizontal spaces $T_z^h S^{2n+1}$ which are the orthogonal complements to the orbits of the S^1-action

$$(z_1, ..., z_{n+1}) \cdot g = (z_1 g, ..., z_{n+1} g).$$

Show: The curvature $F_A \in \Omega^2(\mathbb{C}P^n; \mathrm{i}\mathbb{R})$ defines the symplectic form

$$\omega_{FS} := -\mathrm{i}F_A,$$

Remark: It is called **Fubini Study form**.

Problem 3 [Cotangent Bundle]
(a) Show that the exterior derivative $d\theta$ of the canonical one form $\theta \in \Omega^1(T^*Q)$ on the cotangent bundle of a smooth manifold Q is symplectic form.
(b) Check that the zero section and an arbitrary fibre of T^*Q is a Lagrangian submanifold. Moreover, prove that for any submanifold $R \subset Q$ its **conormal bundle**

$$N(R) := \{\alpha \in T_q^*Q \mid q \in R, \alpha(v) = 0 \forall v \in T_q R\}$$

is a Lagrangian submanifold.
(c) Let $\alpha \in \Omega^1(Q)$ be a smooth one form. Show that its graph

$$\{\alpha_q \mid q \in Q\} \subset T^*Q$$

is Lagrangian if and only if $d\alpha = 0$.

Problem 4 [Conservation Laws]
Show the following claims:
(a) In an autonomous Hamiltonian system (M, ω, H) flow lines lie completely in level sets of H.
(b) Consider a general Hamiltonian system (M, ω, H). Let $\Phi : U \times (t_0 - \epsilon, t_0 + \epsilon) \to M$ be a smooth map for an open subset $U \subset M$ satisfying $\Phi(t_0, x) = x$ for all $x \in U$ and

$$\frac{\partial \Phi}{\partial t}(x, t) = X_H(\Phi(x, t), t)$$

also called the **flow of** X_H. We abbreviate $\Phi_t(X) := \Phi(x, t)$. Then $\Phi_t : U \to M$ is an embedding and

$$\Phi_t^* \omega = \omega.$$

(c) Let $\varphi : (M_1, \omega_1) \to (M_2, \omega_2)$ be a symplectomorphism. Let $H : M_2 \times \mathbb{R} \to \mathbb{R}$ be a smooth function on M_2. Then for the Hamiltonian vector fields of H and $H \circ \varphi$

$$\varphi_*(X_{H\circ\varphi}) = X_H.$$

Problem 5 [Noether's Theorem] Let (M, ω, H) be an autonomous Hamiltonian system. Let G be a Lie group which acts on M such that ω and H is preserved and such that there is a function

$$\varphi : \underline{g} \times M \to \mathbb{R}$$

so that for any $\xi \in \underline{g}$ the Hamiltonian vector field

$$X_{\varphi(\xi,.)} = \tilde{\xi}$$

and the fundamental vector field of ξ agree.
Show that for any $\xi \in \underline{g}$

$$\varphi(\xi, .)$$

is constant along the flowlines of X_H.