# Problem Set 2

## Differential Geometry II
## Summer 2020

**Problem 1**
Prove Theorem 11 (3), that $F^*d\alpha = d(F^*\alpha)$ for $\alpha \in \Omega^k(V)$ and a differentiable map $F$ in a conceptually easier way: $U \subset \mathbb{R}^n \to V \subset \mathbb{R}^m$ between open subsets:
Explain first, how $F^*(df) = d(F^*f)$ for a differentiable function $f \in C^\infty(V)$ follows and use then that $F^*$ is $\mathbb{R}$–linear and respects the wedge-product. (That was suggested by a student in class).

**Problem 2**
(1) For $F; U; V$ as in Problem 1, explain how $F^*$ induces a linear map

$$F^* : H^k_{DR}(V) \longrightarrow H^k_{DR}(U)$$

and show that it is well-defined.
(2) Show that if for two such maps $F, G$ for which there exists a sequence of homomorphisms $P^k : \Omega^k(V) \to \Omega^{k-1}(U)$ such that on $\Omega^k(V)$

$$F^* - G^* = P^{k+1} \circ d + d \circ P^k$$

than $F^* = G^* : H^k_{DR}(V) \longrightarrow H^k_{DR}(U)$.

**Problem 3** [Line integrals]
Let $\gamma : [a, b] \to U$ be a differentiable map, $\alpha \in \Omega^1(U)$ be a differential form. We define the line integral of $\alpha$ over $\gamma$ to be

$$\int_\gamma \alpha := \int_a^b \alpha_{\gamma(t)}(\dot{\gamma}(t))dt.$$

(1) Show that for an oriented reparametrization $\varphi : [c, d] \to [a, b]$ we obtain the same value for the integral of $\alpha$ over $\gamma \circ \varphi : [c, d] \to U$. In fact, try to show that the claim remains valid if $\varphi$ is not necessarily a orientation preserving diffeomorphism but just required to be differentiable with $\varphi(c) = a$ and $\varphi(d) = b$.
(2) Assume that $\gamma$ is a closed curve, i.e. $\gamma(a) = \gamma(b)$. Then

$$\int_\gamma df = 0$$

for any smooth function $f \in C^\infty(U)$.
(3) Let $\alpha \in \Omega^1(U)$ be closed, i.e. $d\alpha = 0$. Let $\Gamma : [a, b] \times [0, 1] \to U$ be differentiable, $\Gamma(a, t) = \Gamma(b, t)$ for all $t$. Then for $\gamma_t := \Gamma(., t) : [a, b] \to U$ the line integrals

$$\int_{\gamma_t} \alpha$$

do not depend on $t$. Hint: Differentiate the integral w.r.t. to $t$.

**Problem 4** [Winding number]
Consider the differential 1–form $\lambda \in \Omega^1(\mathbb{R}^2 \setminus \{0\})$

$$\lambda_{(x,y)} = \frac{xdy - ydx}{x^2 + y^2}.$$

(1) Show that $\lambda$ is closed.
(2) Compute the line integral of $\lambda$ over $\gamma : [0,1] \to \mathbb{R}^2 \setminus \{0\}$, $\gamma(t) = (\cos(2\pi t), \sin(2\pi t))$.
(3) What follows for the class $[\lambda] \in H^1_{DR}(\mathbb{R}^2 \setminus \{0\})$? Solve Problems 13 and 14 of the book of Agricola and Friedrich, page 45.
(4) Conclude that there is no differentiable map $H : [0,1] \times [0,1] \to \mathbb{R} \setminus \{0\}$, such that $H(0,t) = H(1,t)$ for all $t$, $H(s,0) = \gamma(s)$ for all $s$ and $H(.,t)$ is constant.

**Problem 5** [Brouwer's Fixed Point Theorem]
(1) Show that there is no differentiable map $\varphi : D^2 \to S^1$ from the closed unit disk to its boundary which restricts to the identity on $S^1$. Hint: assume it does. Pull back $\lambda$ from Problem 4. Use Problem 3 (3) to conclude that

$$\int_\gamma \lambda = 0$$

for $\gamma$ from Problem 4 (2) and thus a contradiction to the computation there.
(2) Show that every differentiable map $\varphi : D^2 \to D^2$ has a fixed point, by assuming the contrary and constructing a map as described in (1). Hint: Consider the line through $x \in D^2$ and $\varphi(x) \in D^2$. Remark: It follows that the claim is also true for any continuous function but this is not part of the homework problem.)

**Problem 6**
Study problem 3 of the book of Agricola and Friedrich, page 43.