Problem Set 3

Differential Geometry II
Summer 2020

Problem 1

Following discussions about Problem 4 of Problem Set 2 in the tutorial on April 30:

(1) Let $\pi : \mathbb{R} \to \mathbb{R} \setminus \{0\}$ be given by $\pi(s,t) := e^{s+\mathrm{i}t}$. (for those who know this term: this is the universal covering of $\mathbb{R} \setminus \{0\}$). Assume $\alpha \in \Omega^1(\mathbb{R} \setminus \{0\})$ is a closed 1-form, $d\alpha = 0$ and for $\gamma : [0,1] \to \mathbb{R}^2 \setminus \{0\}$, $\gamma(t) = (\cos(2\pi t), \sin(2\pi t))$,

$$\int_\gamma \alpha = 0.$$

Then α is exact, i.e. there exists a smooth function $f : \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}$ with $df = \alpha$. Hint: Consider the pull-back $\pi^*\alpha$. Explain why there is a smooth function $\tilde{f} : \mathbb{R} \to \mathbb{R}$ such that $\pi^*\alpha = d\tilde{f}$. Now, prove that $\tilde{f}(s, t + 2\pi) = \tilde{f}(s,t)$ and conclude the claim. You need to explain that

$$\int_{\gamma_r} \alpha = 0$$

for $\gamma_r : [0,1] \to \mathbb{R}^2 \setminus \{0\}$, $\gamma(t) = (r\cos(2\pi t), r\sin(2\pi t))$ for all $r > 0$.

(2) Discuss Problem 4 (3) of Problem Set 2: The cohomology class of the 1-form λ given there generates $H^1_{DR}(\mathbb{R} \setminus \{0\})$: Pick any closed one-form μ on $\mathbb{R} \setminus \{0\}$ and define

$$a := \frac{1}{2\pi} \int_\gamma \mu \in \mathbb{R}.$$

Show that $[\mu] = a[\lambda] \in H^1_{DR}(\mathbb{R} \setminus \{0\})$.

Problem 2

(1) Determine $H^1_{DR}(S^1)$. The ideas of Problem 1 could be helpful. Since $\dim S^1 = 1$ it is somewhat simpler.

(2) Hard: Determine $H^1_{DR}(S^n)$ "by hand" without technology from algebraic topology.

Problem 3

Let M be a manifold without boundary, $f : M \to \mathbb{R}$ a smooth function, $c \in \mathbb{R}$ a regular value, i.e. for all $p \in M$, $f(p) = c$ we have $d_p f \neq 0$. Then the sublevel set

$$M^c := \{p \in M \mid f(p) \leq c\}$$

is a smooth manifold with boundary: $\partial M^c = \{p \in M \mid f(p) = c\}$.

Problem 4

Show that the boundary of a smooth n–dimensional manifold with boundary is an $(n-1)$–dimensional manifold without boundary.

Problem 5

For some of you that problem is almost a repetition from these facts for manifolds without boundary.

(1) Explain how the tangent space of a manifold with boundary ata point p is a real vector space.

(2) Show that an inward pointing vector cannot be outward pointing.

(3) Show that the differential af a smooth map between manifolds with boundary is a linear map between vector spaces.

Problem 6

Prove Lemma 24 of the lecture:Let M be an oriented manifold with boundary of dimension $m \geq 2$. Then the tangent spaces at all boundary points can be oriented so that there exists a chart around each which is oriented in the sense of Definition 23. In particular, the boundary can be oriented so that for any $p \in \partial M$ an oriented basis of $T_p(\partial M)$ extended by an **outward** pointing tangent vector put in the **first position** gives an oriented basis of T_pM.