# Problem Set 4

## Differential Geometry II
## Summer 2020

**Problem 1**
(a) Step (1) in Proof of Brouwer's fixed point theorem: Suppose $F : B^n \to B^n$ is a smooth map without fixed points. Let $f : B^n \to S^{n-1}$ be the map which assigns to $x \in B^n$ the intersection of the line through $x$ and $F(x)$ with $S^{n-1}$ such that the segment $[f(x), F(x)]$ contains $x$. Explain why that $f$ is smooth.
(b) Show how any continuous map $F : B^n \to B^n$ can be approximated uniformly by a sequence of smooth maps $F_n : B^n \to B^n$.
(c) Let $(x_n)$ be a sequence of fixed points of $F_n$ from (2) which converges. Show that the limit is a fixed point of $F$.

**Problem 2**
Show that a convex linear combination of positive definite symmetric bilinear forms is also a positive definite symmetric bilinear form.

**Problem 3**[Volume Form]
Let $(M, g)$ be an oriented Riemannian manifold with boundary.
(a) Let $(U, \varphi, V)$ be an oriented coordinate chart, $g = g_{ij}$ the Gram matrix w.r.t. the coordinate vector fields. Show that

$$\varphi^*(dM) = \sqrt{\det(g)}dx^1 \wedge ... \wedge dx^n.$$

(b) Recall that $\mathbf{n}_p$ denotes the outer normal vector field, i.e. the outward tangent vector at $p \in \partial M$ with $\mathbf{n}_p \perp T_p(\partial M)$. Show that the two volume forms of $M$ and $\partial M$ are related by

$$d(\partial M) = \mathbf{n}_p \lrcorner dM.$$

**Problem 4**[Classical Differential Operators]
(a) Setting as in Problem 3. For the Laplacian on functions prove in local coordinates

$$\Delta f = -\frac{1}{\sqrt{\det g}} \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\Big(\sqrt{\det g} g^{ij} \frac{\partial f}{\partial x_j}\Big).$$

(b) Determine the local expression of $\mathrm{div}X$ for a vector field $X$.
(c) For a smooth function $f$ denote by $\nabla f$ the vector field defined by

$$g(\nabla f, .) = df.$$

Show Green's formulas: For two smooth functions $f, g$ on $M$

$$\int_M \langle \Delta f, g\rangle + \langle \nabla f, \nabla g\rangle)dM = \int_{\partial M} \langle g\nabla f, \mathbf{n}\rangle d(\partial M)$$

and

$$\int_M \langle \Delta f, g\rangle - \langle f, \Delta\rangle)dM = \int_{\partial M} \langle g\nabla f - f\nabla g, \mathbf{n}\rangle d(\partial M).$$

**Problem 5**[Classical Stokes' Theorem]
We consider $M = \mathbb{R}^3$ with the standard euclidean structure.
(a) The **Rotation** of a vector field $X$ is the vector field defined by

$$\mathrm{rot}X = \begin{pmatrix} \frac{\partial X_2}{\partial x_3} - \frac{\partial X_3}{\partial x_2} \\ -\frac{\partial X_1}{\partial x_3} + \frac{\partial X_3}{\partial x_1} \\ \frac{\partial X_1}{\partial x_2} - \frac{\partial X_2}{\partial x_1} \end{pmatrix}$$

Show the following identities

$$\begin{aligned} \mathrm{rot} \circ \nabla &= 0 \\ \mathrm{div} \circ \mathrm{rot} &= 0 \\ \nabla \circ \mathrm{div} \pm \mathrm{rot} \circ \mathrm{rot} &= -\Delta \end{aligned}$$

Determine the correct signs in the last equations.
(b) As a vector field $X$ determines a 1–form on $\mathbb{R}^3$ via $X^* := \langle X, . \rangle$ it also defines a 2–form via $\alpha_X(Y, Z) := \langle X, Y \times Z \rangle$ using the cross– or vector product on $\mathbb{R}^3$. Interpret $\nabla, \mathrm{div}, \mathrm{rot}$ on functions and vector fields in the language of differential forms. To what identities do the relations above boil down?
(c) Let $X$ be a vector field on $\mathbb{R}^3$ and $F \subset \mathbb{R}^3$ be an oriented smooth surface with boundary. Let $N : F \to \mathbb{R}^3$ denote the normal field to $F$ such that an oriented basis of $T_pF$ extenden by $N_p$ in the first position gives a standard oriented basis of $\mathbb{R}^3$. Prove

$$\int_F \langle \mathrm{rot}X, N \rangle dF = \int_{\partial M} \langle X, d(\vec{\partial} F) \rangle$$

where $d(\vec{\partial} F)$ denotes the vector valued one form which in a local parametrization$\gamma : I \to F$ of $\partial F$ is given by $\dot{\gamma(t)} dt$.