Problem Set 5

Differential Geometry II
Summer 2020

Problem 1 [Examples]
(a) Recall the definition of the Möbius strip and show that it is a fibre bundle.
(b) Show that the Hopf bundle is a fibre bundle of manifolds (see lecture on "Fibre Bundles" for a description)
(c) Construct a metric on TM with the help of a Riemannian metric on M.

Problem 2 [Pull-Back Bundles and Connections]
(a) Let (E, B, π, F) be a topological fibre bundle, $\varphi : C \to B$ be a continuous map. Show that there is a topological fibre bundle (φ^*E, C, p, F) together with a bundle morphism $\Phi : \varphi^*E \to E$ covering φ which is a homeomorphism on each fibre. Show that this bundle together with the bundle map is unique up to an isomorphism (covering the identity on C) which commutes with the bundle morphisms. Hint: The construction via cocycles (see Lemma 43) is convenient.
(b) Does the statement remain true for smooth fibre bundles of manifolds?
(c) Let ∇ be a connection on a vector bundle E over a manifold M. Let $\varphi : N \to M$ be a smooth map between manifolds. Prove that there exists a unique connection ∇^φ on φ^*E satisfying

$$\nabla^\varphi_X(\sigma \circ \varphi) = \nabla_{d_p\varphi(X)}\sigma$$

for any smooth section σ of E and $X \in T_pN$.

Problem 3 [The Hedgehog Theorem]
(a) Let M be an orientable manifold. Show that an orientation–reversing diffeomorphism cannot be homotopic through smooth maps to a orientation–preserving one. How could we even exclude a continuous homotopy?
(b) Show that the antipodal map $x \mapsto -x$ on a sphere S^n is homotopic to the identity if and only if n is odd.
(c) Show that a sphere S^{2k} does not admit a nowhere vanishing vector field. Hint: Assuming it does, construct a homotopy between the identity and the antipodal map.
(d) Construct such a vector field for S^3. Can you describe such a vector field for any odd–dimensional sphere?

Problem 4 [The Tautological Line Bundle]
(a) Show that

$$H := \{([z_1, z_2], (\lambda z_1, \lambda z_2)) | [z_1, z_2] \in \mathbb{C}P^1, \lambda \in \mathbb{C}\}$$

is a smooth vector bundle. Describe a trivialization on

$$U_k := \{[z_1, z_2] \in \mathbb{C}P^1 | z_k \neq 0\}$$

for $k = 1, 2$ and the corresponding transition function. Can you explain, why this bundle must be non-trivial?
(b) A section $\sigma : U \to H|_U \subset U \times \mathbb{C}^2$ can be considered as two complex funtions. We define a connection ∇ on the bundle by

$$\nabla\sigma := \mathrm{proj}^\perp_H(d\sigma)$$

where $\mathrm{proj}^\perp_H$ denotes the orthogonal projection w.r.t. the standard scalar product. Show that this is a connection which satifies Leibniz' Rule even for complex valued smooth functions. Express it in the trivializations found in (a). Hint: Make use of the Hermitian product on $\mathbb{C}^2$.

Problem 5 [Euler Field and Tautological 1–Form]
(a) Let $E \xrightarrow{\pi} M$ be a vector bundle over a manifold M. We define the **Euler field** X on TE by

$$X(e) := e \in T_e E_{\pi(e)}.$$

Determine the flow of this vector field.
(b) Let M be a smooth manifold. The 1–form $\theta \in \Omega^1(T^*M)$ is defined to be

$$\theta_\alpha(X) = \alpha(d_\alpha \pi^*(X))$$

where $X \in T_\alpha(T^*M)$ and $d\pi^*$ is the differential of $\pi^* T^*M \to M$. Show that θ is indeed smooth and compute its differential.
For a smooth curve $\gamma : [a,b] \to M$ in a Riemannian manifold (M,g) we define $\alpha : [a,b] \to T^*M$ via

$$\alpha(t) := g_{\gamma(t)}(\dot\gamma(t), .).$$

Compare the integral

$$\int_a^b \alpha^*\theta$$

with something familiar.

Problem 6 [Alternative Definition of a Connection]
Let $E \xrightarrow{\pi} M$ be a vector bundle over the manifold M.
(a) Show that for the fibrewise product the maps

$$\alpha : (v,w) \in E_\pi \times_\pi E \to (v+w) \in E$$

and

$$\mu : (\lambda, v) \in \mathbb{R} \times E \to \lambda v \in E$$

are smooth.
(b) Can you show the contrary? Let (E, B, π, F) be a fibre bundle of manifolds, so that every fibre is a real vector space such that the corresponding maps described in (a) are smooth. Then it is a smooth vector bundle.
(c) Show that a connection on a vector bundle gives rise to a splitting of the tangent spaces to its total space

$$T_e E = T_e^h E \oplus E_p.$$

In particular $d\pi|_{T_e^h E}$ is an isomorphism. It satisfies the following condition: Let $(e_1, e_2) \in E \times E$ and $T^h_{e_1,e_2} E \times E = T^h_{e_1} E \oplus T^h_{e_2} E$. For $e_1, e_2 \in \pi^{-1}(p)$ we define

$$T^h_{e_1,e_2}(E \oplus E) := (d_{(e_1,e_2)}\pi|_{T^h_{e_1} E \oplus T^h_{e_2} E})^{-1}(T_p \Delta_M).$$

on the direct sum where $\Delta_M \subset M \times M$ denotes the diagonal. Then

$$d_{e_1,e_2}\alpha(T^h_{e_1,e_2}(E \oplus E)) = T^h_{e_1+e_2} E,$$

and

$$d_{(\lambda,e)}\mu(\{0\} \oplus T_e^h E) = T^h_{\lambda e} E.$$

(d) A splitting of TE as in (c) defines a connection: If $\sigma : U \to E|_U$ is a section then

$$(\nabla\sigma)(p) := \mathrm{pr}_{E_p}(d\sigma)$$

where pr_{E_p} is the projection with respect to the splitting of $T_e E = T_e^h E \oplus E_{\pi(e)}$, is a connection on the vector bundle $E \xrightarrow{\pi} E$. Hint: You need to show that for the projections w.r.t. the splittings

$$\mathrm{pr}_{T^h(E \oplus E)(V,W)} = (\mathrm{pr}_{T^h E}(V), \mathrm{proj}_{T^h E}(W))$$

for $V \in T_v E, W \in T_w E$ with $v, w \in \pi^{-1}(p)$ and $d_v\pi(V) = d_w\pi(W)$.