Problem Set 6

Differential Geometry II
Summer 2020

Problem 1 [Curvature]
Show (2) and (3) of Proposition 57: (a) Let ∇^0, ∇ be two connections, $\nabla = \nabla^0 + \alpha$, for $\alpha \in \Omega^1(M; End(E))$. Then with $D^0 : \Omega^1(M; End(E)) \to \Omega^2(M; End(E))$

$$F^{\nabla} = F^{\nabla^0} + D^0\alpha + \alpha \wedge \alpha.$$

(b) Let $p \in M$, $e \in E_p$, X, Y be two vector fields on M in a neighbourhood of p. Let $\tilde{X}, \tilde{Y}$ be their horizontal lifts to E, $\tilde{X}_e = (d_e\pi|_{T^heE})(X_{\pi(e)})$. Then

$$F^{\nabla}(X,Y)e = [\tilde{X}, \tilde{Y}]_e - \widetilde{[X,Y]}_e$$

(Note: This has been corrected!)
(c) Show that the curvature of a metric connection is skew-symmetric. (property (3) of the Remark after Definition 59).
Explain why $A^i_j = -A^j_i$ for the connection-1–form and $F^i_j = -F^j_i$ for the curvature w.r.t. a euclidean trivialization.

Problem 2 [Pull-Back Bundle, Connections and Curvature]
(a) Let (E, B, π, F) be a topological fibre bundle, $\varphi : C \to B$ be a continuous map. Show that there is a topological fibre bundle (φ^*E, C, p, F) together with a bundle morphism $\Phi : \varphi^*E \to E$ covering φ which is a homeomorphism on each fibre. Show that this bundle together with the bundle map is unique up to an isomorphism (covering the identity on C) which commutes with the bundle morphisms. Hint: The use of the construction via cocycles (see Lemma 43) is convenient.
(b) Does the statement remain true for smooth fibre bundles of manifolds?
(c) Let ∇ be a connection on a vector bundle E over a manifold M. Let $\varphi : N \to M$ be a smooth map between manifolds. Prove that there exists a unique connection ∇^{φ} on φ^*E satisfying

$$\nabla^{\varphi}_X(\sigma \circ \varphi) = \nabla_{d_p\varphi(X)}\sigma$$

(d) How are the curvatures of ∇ and ∇^{φ} related?

Problem 3 [The Tautological Line Bundle]
(a) Show that

$$H := \{([z_1, z_2], (\lambda z_1, \lambda z_2)) | [z_1, z_2] \in \mathbb{C}P^1, \lambda \in \mathbb{C}\}$$

is a smooth vector bundle. Describe a trivialization on

$$U_k := \{[z_1, z_2] \in \mathbb{C}P^1 | z_k \neq 0\}$$

for $k = 1, 2$ and the corresponding transition function. Can you explain, why this bundle must be non-trivial?
(b) A section $\sigma : U \to H|_U \subset U \times \mathbb{C}^2$ can be considered as two complex funtions. We define a connection ∇ on the bundle by

$$\nabla\sigma := \mathrm{proj}^{\perp}_H(d\sigma)$$

where $\mathrm{proj}^{\perp}_H$ denotes the orthogonal projection w.r.t. the standard scalar product. Show that this is a complex connection, i.e. satifies Leibniz' Rule even for complex valued smooth functions. Express it in the trivializations found in (a). Hint: Make use of the Hermitian product on $\mathbb{C}^2$.
(c) Compute the curvature of ∇. Explain that it is a 2–form on $\mathbb{C}P^1$ (with purely imaginary values).

Problem 4 [Tautological 1–Form]
Let M be a smooth manifold. The 1–form $\theta \in \Omega^1(T^*M)$ is defined to be

$$\theta_\alpha(X) = \alpha(d_\alpha \pi^*(X))$$

where $X \in T_\alpha(T^*M)$ and $d\pi^*$ is the differential of $\pi^* T^*M \to M$. Show that θ is indeed smooth and compute its differential.
For a smooth curve $\gamma : [a,b] \to M$ in a Riemannian manifold (M,g) we define $\alpha : [a,b] \to T^*M$ via

$$\alpha(t) := g_{\gamma(t)}(\dot{\gamma}(t), .).$$

Compare the integral

$$\int_a^b \alpha^*\theta$$

with something familiar.