Problem Set 8

Differential Geometry II
Summer 2020

Problem 1 [Exterior Covariant Derivative]

(a) Let $A \in \Omega^1(P, \underline{g})$ be a connection of the principal G–bundle $P \xrightarrow{\pi} M$. Recall that

$$\Omega^k(M; \underline{\mathbf{g}}) = \{\alpha \in \Omega^k(P, \underline{g}) | \forall X \in \underline{g} : \tilde{X} \lrcorner \alpha = 0, \mu_g^* \alpha = Ad_{g^{-1}} \alpha\}$$

Show that

$$D_A \alpha = d\alpha + [A, \alpha]$$

defines a linear map $D_A : \Omega^k(M; \underline{\mathbf{g}}) \to \Omega^{k+1}(M\underline{\mathbf{g}})$.

Note that for $A = \sum_{i=1}^n A_i dx^i$ $A_i \in \underline{g}$ and $\alpha = \sum_I \alpha_I dx^I$ where $I = 1 \le i_1 < i_2 < ... < i_k \le n$ is a multi-index, $dx^I = dx^{i_1} \wedge ... \wedge dx^{i_k}$ and $\alpha_I \in \underline{g}$

$$[A, \alpha] = \sum_{i=1}^n \sum_I [A_i, \alpha_I] dx^i \wedge dx^I.$$

(b) Show that $F_A \in \Omega^2(M; \underline{\mathbf{g}})$, i.e. repeat the proof that $\tilde{X} \lrcorner F_A = 0$ and show $\mu_g^* F_A = Ad_{g^{-1}} F_A$.

Problem 2 [Hopf Bundle]
This is a repetition of what was explained in class. Let

$$S^3 := \{(z_1, z_2) \in \mathbb{C}^2 \mid |z_1|^2 + |z_2|^2 \ = 1\} \subset \mathbb{C}^2.$$

The Lie group $S^1 = U(1) = SO(2)$ is acting on it (from the right) via $z \mapsto zg$.
(a) Construct a homeomorphism

$$S^3/S^1 \cong S^2$$

such that the quotient map $S^3 \xrightarrow{\pi} S^2$ is smooth.
(b) Show that $S^3 \xrightarrow{\pi} S^2$ is a principal S^1–bundle.
(c) Verify that the orthogonal complements of the fibre tangents, $T_p^h S^7 := (T_p \pi^{-1}([p]))^\perp$, define a connection of the principal S^1–bundle.
(d) Describe the curvature of this connection (e.g. in local charts). Determine the Chern classes of the Hopf bundle, i.e. the Chern classes of the associated vector bundle w.r.t. the representation $S^1 \to \mathbb{C}^*$.

Problem 3 [Tautological Line Bundle]
Once again: (a) Show that

$$H := \{([z_1, z_2], (\lambda z_1, \lambda z_2)) | [z_1, z_2] \in \mathbb{C}P^1, \lambda \in \mathbb{C}\}$$

is a smooth vector bundle. Describe a trivialization on

$$U_k := \{[z_1, z_2] \in \mathbb{C}P^1 | z_k \neq 0\}$$

for $k = 1, 2$ and the corresponding transition function. Can you explain, why this bundle must be non-trivial?
(b) A section $\sigma : U \to H|_U \subset U \times \mathbb{C}^2$ can be considered as two complex funtions. We define a connection ∇ on the bundle by

$$\nabla\sigma := \text{proj}_H^{\perp}(d\sigma)$$

where $\text{proj}_H^{\perp}$ denotes the orthogonal projection w.r.t. the standard scalar product. Show that this is a complex connection, i.e. satifies Leibniz' Rule even for complex valued smooth functions. Express it in the trivializations found in (a). Hint: Make use of the Hermitian product on $\mathbb{C}^2$.
(c) Compute the curvature of ∇. Explain that it is a 2–form on $\mathbb{C}P^1$ (with purely imaginary values).
(d) Consider the Hopf bundle and the representation $\rho : S^1 \to Aut(\mathbb{C}) = \mathbb{C} \setminus \{0\}$ given by $\rho(g) = g$. Show that

$$S^3 \times_\rho \mathbb{C} \cong H.$$

Problem 4 [Invariant Polynomials and Chern Classes]
(a) Recall that we defined in class

$$s_\ell(E, \nabla) = \text{Trace}\Big((F^\nabla)^\ell\Big)$$

where the power means the ℓ-fold Wedge product of F^∇ with itself. Express the Chern Classes in terms of s_ℓ for the first $c_1, c_2, c_3, ...$
(b) Find a general formula or prove that each c_k is a polynomial of s_ℓ.