Problem Set 9

Differential Geometry II
Summer 2020

Problem 1 [Exterior Derivatives]
(a) Let $\omega \in \Omega^k(M)$ be a k-form on the manifold M, $X_1, ..., X_{k+1}$ vector fields. Prove the following identity:

$$d\omega(X_1, ..., X_{k+1}) = \sum_{j=1}^{k+1} (-1)^{j-1} X_j(\omega(X_1, ..., \widehat{X_j}, ..., X_{k+1})) + \sum_{1 \le i < j \le k+1} [X_i, X_j](\omega([X_i, X_j], X_1, ..., \widehat{X_i}, ..., \widehat{X_j}, ..., X_{k+1})).$$

(b) Let ∇ be a covariant derivative on a vector bundle $E \xrightarrow{\pi} M$. $\omega \in \Omega^k(M; E)$ and $X_1, ..., X_{k+1}$ vector fields on M. Guess a formula for the exterior covariant derivative $D\omega$ similar to (a) and prove it. Derive a formula for the curvature $F(v, w)$ in terms of ∇ using vector field extensions of the tangent vectors $v, w \in T_pM$.

Problem 2 [Cartan's (Magic) Formula]
(a) Let X, Y be vector fields on a manifold M (w.l.o.g. M is an open subset of $\mathbb{R}^n$ since we are dealing with a local problem). Let $p \in M$ and $\Phi : (-\epsilon, \epsilon) \times V \to M$ be the flow map defined on a neighbourhood V of p and some $\epsilon > 0$ (V and ϵ depending on V always exist!). Prove that

$$\frac{d}{dt}\Big|_{t=0} d_{\Phi_t(p)}\Phi_{-t}(Y_{\Phi_t p}) = [X, Y]_p,$$

where $\Phi_t : V \to M$ is defined to by $\Phi_t(p) = \Phi(t, p)$. That expression is defined to be the **Lie derivative** $\mathcal{L}_X Y$ of Y along X.
(b) For a differential form $\omega \in \Omega^k(M)$ we define

$$(\mathcal{L}_X\omega)_p := \frac{d}{dt}\Big|_{t=0} (\Phi_t^*\omega)_p.$$

Show Cartan's magic formula

$$\mathcal{L}_X\omega = X \lrcorner d\omega + d(X \lrcorner \omega).$$

Problem 4 [$SU(k)$–Bundles]
(1) Let $E \xrightarrow{\pi} M$ be a Hermitian vector bundle of complex rank k, $P \to M$ its unitary frame bundle. Assume that there exists a reduction $Q \to P$ of P w.r.t. the inclusion homomorphisms $SU(k) \hookrightarrow U(k)$. What is the underlying differentiable structure on E. Hint: Recall that $SU(k) = \{A \in U(k) \mid \det A = 1\}$.
(2) Show that a reduction of the unitary frame bundle as in (i) exists if and only if $c_1(E) = 0$.

Problem 5 [Arithmetic of Chern Classes]
Let $L_k \xrightarrow{\pi} M$ be two complex line bundles over a manifold M, i.e. complex vector bundles of complex rank 1. Show the folllowing formulas for Chern classes

$$c_1(L_1 \oplus L_2) = c_1(L_1) + c_1(L_2), \quad c_2(L_1 \oplus L_2) = c_1(L_1)c_1(L_2), \quad c_1(L_1 \otimes L_2) = C_1(L_1) + c_1(L_2).$$

Recall, that the product on $H^*_{DR}(M)$ is the induced by the wedge-product on $\Omega^*(M)$. You should use the Chern-Weil forms for the classes to prove the statements.

Problem 6s [Quaternionic Hopf Bundle]
Let

$$S^7 := \{A \in M(2;\mathbb{C}) \mid \mathrm{Trace}(\overline{A}^T A) = 2\} \subset M(2;\mathbb{C}) \cong \mathbb{C}^4.$$

The Lie group $SU(2) := \{g \in M(2;\mathbb{C}) | \overline{g}^T g = \mathbf{E}_2, \det g = 1\}$ is acting on it (from the right) via $A \mapsto Ag$.
(a) Construct a homeomorphism

$$S^7/SU(2) \cong S^4$$

such that the quotient map $S^7 \xrightarrow{\pi} S^4$ is smooth.
(b) Show that $S^7 \xrightarrow{\pi} S^4$ is a principal $SU(2)$–bundle.
(c) Verify that the orthogonal complements of the fibre tangents, $T_p^h S^7 := (T_p \pi^{-1}([p]))^\perp$, define a connection of the principal $SU(2)$–bundle.
(d) Compute the curvature of this connection. Determine the Chern classes of the Quaternionic Hopf bundle, i.e. the Chern classes of the associated complex vector bundle of rank 2 w.r.t. the representation $SU(2) \to Gl(2;\mathbb{C})$.