Differential Geometry II

Curvature

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The Space of Connections

$E \to M$ vector bundle

$E_p = \pi^{-1}(p)$

The set of all connections

$$\mathcal{C}(E) = \{\nabla | \nabla \text{ connection of } E\}$$

is an affine space over

$$\alpha \in \Omega^1(M; End(E)) = \Gamma(T^*M \otimes End(E))$$
$$= \{\sigma : M \to T^*M \otimes \underline{E^* \otimes E} \mid \sigma \text{ smooth section}\}.$$

if $X \in T_pM$ $\quad \alpha_p(X) \in End(E_p)$

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Consider $\alpha := \nabla - \nabla^0 : \Gamma(E) \to \Omega^1(E)$.

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$\sigma : M \to E$ section, smooth
$f \in C^\infty(M)$

$$\alpha(f\sigma) = \nabla(f\sigma) - \nabla^0(f\sigma) = f(\nabla\sigma - \nabla^0\sigma) + (df)\sigma - (df)\sigma = f\alpha(\sigma)$$

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For $v \in E_p$ let $\sigma \in \Gamma(U, E|_U)$ sich that $\sigma(p) = v$ and define

$$\alpha_p(v) := \alpha(\sigma)(p).$$

Pull-Backs

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$$g^*E = \coprod_{p \in P} E_{g(p)} \xrightarrow{\pi} P$$

where a trivialization $\Phi : \pi^{-1}(U) \to U \times \mathbb{R}^k$ of E over $U \subset M$ induces a trivialization $\Phi_g : g^{-1}(\pi^{-1}(U)) \to g^{-1}(U) \times \mathbb{R}^k$ via

$$\Phi_g(e) = (p, \mathrm{pr}_{\mathbb{R}^k}\Phi(e)) \qquad \subset P \text{ open}$$

for $e \in (g^*E)_p$. $= E_{g(p)}$ $\quad$ ($p_1 \neq p_2$ & $g(p_1) = g(p_2)$)

$(g^*E)_{p_1} \neq (g^*E)_{p_2}$

$\{U_i\}_i$ covering of M $\Rightarrow$ $\{g^{-1}(U_i)\}_i$ covering of P

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(2) The **pull back**, ∇^g, of the connection ∇ is given w.r.t. the trivialization by the connection 1–form

$$A_\Phi^g := g^* A_\Phi.$$

... pull-back of 1-forms

Parallel Transport

∇^g is well–defined, i.e. independent of the local trivialization Φ of E.

If for two trivializations Φ, Ψ

$$\Psi \circ \Phi^{-1}(x, v) = (x, \varphi(x) v)$$

$$\varphi : U \cap V \to Gl(k, \mathbb{R}) \quad \text{transition function}$$

$$\Leftrightarrow \quad \varphi \circ g : g^{-1}(U) \cap g^{-1}(V) \to Gl(k, \mathbb{R})$$

transition functions for triv. of g^*E

$$A^g_\Psi = (\varphi \circ g)^{-1} A^g_\Phi (\varphi \circ g) + (\varphi \circ g)^{-1} d(\varphi \circ g)$$

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σ is called **horizontal lift** of γ or just **horizontal curve**.

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(ii) A **horizontal splitting** $T_eE = T_e^hE \oplus E_{\pi(e)}$ which depends smoothly on e and satisfies for $\mu_\lambda : E \to E$, $\mu_\lambda(e) = \lambda e$

$$d_e\mu_\lambda(T_e^hE) = T_{\lambda e}^hE.$$

Proof: (i)$\Rightarrow$ (ii): Given a covariant derivatve, we define for $e \in E_p$

$$\begin{aligned} T_e^hE &:= \{\dot{\sigma}(0) \mid \sigma : I \to E, 0 \in I, \text{ horizontal}\} \\ &= \{(d_e\Phi)^{-1}(X, v) \mid X \in T_pU, v \in \mathbb{R}^k, A_{\Phi,p}(X)\Phi(e) + v = 0\} \subset T_eE \end{aligned}$$

for a trivialization $\Phi : \pi^{-1}(U) \to U \times \mathbb{R}^k$ and the connection 1–form A_Φ of ∇ w.r.t. Φ.

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(ii)⇒(i): ∇ defined via

$$(\nabla\sigma)_p := \mathrm{pr}_{E_p} d_p \sigma$$

on sections $\sigma : U \to E|_U$ where pr_{E_p} is the projection with respect to the splitting is a covariant derivative whose horizontal splitting is the given one.

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$$D_{k+\ell}(\alpha \wedge \sigma) = d\alpha \wedge \sigma + (-1)^k \alpha \wedge D_\ell \sigma$$

for any $\alpha \in \Omega^k(M)$ and $\sigma \in \Omega^\ell(M, E)$.

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In particular, we have

$$F^\nabla(X,Y)\sigma = (\nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]})\sigma.$$

for any $\sigma \in \Gamma(E)$ and vector fields X, Y.

Curvature

F^{∇} is the **curvature of** ∇.

Proposition 57: (1) Let $A \in \Omega^1(U; M(n;\mathbb{R}))$ be the connection 1-form w.r.t. a trivialization. Then for the curvature we have

$$F_A = dA + A \wedge A \in \Omega^2(U; M(n;\mathbb{R}))$$

w.r.t. the trivialization. Hereby with $A = (A^i_j) in \Omega^1(U))$

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(3) Let $p \in M$, $e \in E_p$, X, Y be two vector fields on M in a neighbourhood of p. Let $\tilde{X}, \tilde{Y}$ be their horizontal lifts to E. Then

$$F^\nabla(X, Y)e = [\tilde{X}, \tilde{Y}]_p.$$

Curvature

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Proof:

Curvature

2nd Bianchi Identity

Proposition 58: With the notation from above we have

$$DF^{\nabla} = 0.$$

Proof:

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A **metric connection** on a euclidean vector bundle (E, g) is a covariant derivative ∇ which satisfies in addition

$$d(g(\sigma, \tau)) = g(\nabla\sigma, \tau) + g(\sigma, \nabla\tau)$$

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Remark: (1) The metric condition is much harder to define in terms of the horizontal vector spaces of TE.
(2) The parallel transport of a metric connection defines isometries between the fibres.
(3) The curvature F of a metric connection is skew-symmetric:

$$g(F(e), f) = -g(e, F(f)).$$

Euclidean Vector Bundles

A euclidean vector bundle can be locally trivialised by isometries:

$$\Phi : \pi^{-1}(U) \to U \times \mathbb{R}^k$$

such that

$$\Phi|_{E_p} : (E_p, g_p) \longrightarrow (\mathbb{R}^k, \langle ., . \rangle)$$

is an isometry for all $p \in U$. Φ will be called **euclidean trivialization**.

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In particular, the transition functions are smooth maps

$$g : U \cap V \to O(k)$$

to the set of orthogonal matrices.

Euclidean Vector Bundles

An **oriented** vector bundle is a choice of orientations of all E_p such that the trivializations Φ can be chosen, so that

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Vice versa: A family of transition funtions

$$g_{ij} : U_i \cap U_j \to O(k) \text{ or } SO(k)$$

for an open covering $\{U_i\}_{i \in I}$ satsifying the cocycle condition defines an (oriented) euclidean vector bundle over M up to (orientation) and metric preserving ismomorphisms (short: isometries).

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One defines for $A = \sum_i A_i dx^i$ and $B = \sum_i B_i dx^i$

$$[A, B] = [A \wedge B] = \sum_{i,j} [A_i, B_j] dx^i \wedge dx^j$$

with $[X, Y] = XY - YX$.

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where $\underline{o}(n) \subset M(k; \mathbb{R})$ denotes the set of skew–symmetric matrices.

Note: For $A, B \in \Omega^1(U; \underline{o}(n))$ in general $A \wedge B \notin \underline{o}(n)$ but $A \wedge A \in \Omega^2(U; \underline{o}(n))$.

One defines for $A = \sum_i A_i dx^i$ and $B = \sum_i B_i dx^i$

$$[A, B] = [A \wedge B] = \sum_{i,j} [A_i, B_j] dx^i \wedge dx^j$$

with $[X, Y] = XY - YX$. Then $A \wedge A = \frac{1}{2}[A, A]$.

Complex Vector Bundles

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Vice versa: A family of such transition functions satisfying the cocycle condition defines a complex vector bundle up to isomorphism.