Differential Geometry II

Euclidean, Complex and Hermitian Structures

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Curvature

Let ∇ be a connection on the vector bundle $E \xrightarrow{\pi} M$ rk $E = k$

Proposition 57: (1) Let $A \in \Omega^1(U; M(n; \mathbb{R}))$ be the connection 1-form w.r.t. a trivialization. Then for the curvature we have

$$F^\nabla =: F_A = dA + A \wedge A \in \Omega^2(U; M(k; \mathbb{R}))$$

w.r.t. the trivialization. Hereby with $A = (A^i_j) \in \Omega^1(U))$

$$(A \wedge A)^i_j = \sum_{\ell=1}^{k} A^i_\ell \wedge A^\ell_j.$$

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$$(A \wedge A)^i_j = \sum_{\ell=1}^{k} A^i_\ell \wedge A^\ell_j.$$

(2) Let ∇^0, ∇ be two connections, $\nabla = \nabla^0 + \alpha$, for $\alpha \in \Omega^1(M; End(E))$. Then

$$F^\nabla = F^{\nabla^0} + D^0\alpha + \alpha \wedge \alpha.$$

∇ connection on $E \rightsquigarrow \nabla$ induces connection on $End(E)$:

if $\sigma \in \Gamma(E)$ section, $\phi \in \Gamma(End(E))$

$(\nabla_X \phi)(\sigma) := \nabla_X(\phi(\sigma)) - \phi(\nabla_X \sigma)$

$\rightsquigarrow$ exterior covariant derivatives $D : \Omega^k(M, End(E)) \to \Omega^{k+1}(M, End(E))$

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(3) Let $p \in M$, $e \in E_p$, X, Y be two vector fields on M in a neighbourhood of p. Let $\tilde{X}, \tilde{Y}$ be their horizontal lifts to E, $\tilde{X}_e = (d_e\pi|_{T^heE})^{-1}(X_{\pi(e)})$. Then

$\pi : E \to M$

$d_e\pi|_{T^h_eE} : T^h_eE \xrightarrow{\cong} T_pM$

$$F^{\nabla}(X, Y)e = [\tilde{X}, \tilde{Y}]_e.$$

Curvature

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Proof of Proposition 57:

Curvature

2nd Bianchi Identity

Proposition 58: With the notation from above we have

$$DF^{\nabla} = 0.$$

Proof:

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Definition 59: Let $E \xrightarrow{\pi} M$ be a smooth vector bundle. A euclidean structure on E is a smooth family $\{g\}_{p \in M}$ of scalar products on the fibres E_p.

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A **metric connection** on a euclidean vector bundle (E, g) is a covariant derivative ∇ which satisfies in addition

$$d(g(\sigma, \tau)) = g(\nabla\sigma, \tau) + g(\sigma, \nabla\tau)$$

for any pair of (local) sections σ, τ of E.

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(2) The parallel transport of a metric connection defines isometries between the fibres.

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Remark: (1) The metric condition is much harder to define in terms of the horizontal vector spaces of TE.
(2) The parallel transport of a metric connection defines isometries between the fibres.
(3) The curvature F of a metric connection is skew-symmetric:

$$g(F(e), f) = -g(e, F(f)).$$

Euclidean Vector Bundles

A euclidean vector bundle can be locally trivialised by isometries:

$$\Phi : \pi^{-1}(U) \to U \times \mathbb{R}^k$$

such that

$$\Phi|_{E_p} : (E_p, g_p) \longrightarrow (\mathbb{R}^k, \langle ., . \rangle)$$

is an isometry for all $p \in U$. Φ will be called **euclidean trivialization**.

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In particular, the transition functions are smooth maps

$$g : U \cap V \to O(k)$$

to the set of orthogonal matrices.

Euclidean Vector Bundles

An **oriented** vector bundle is a choice of orientations of all E_p such that the trivializations Φ can be chosen, so that

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Vice versa: A family of transition funtions

$$g_{ij} : U_i \cap U_j \to O(k) \text{ or } SO(k)$$

for an open covering $\{U_i\}_{i \in I}$ satsifying the cocycle condition defines an (oriented) euclidean vector bundle over M up to (orientation) and metric preserving ismomorphisms (short: isometries).

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with $[X, Y] = XY - YX$.

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with $[X, Y] = XY - YX$. Then $A \wedge A = \frac{1}{2}[A, A]$ and $F = dA + \frac{1}{2}[A, A]$.

Complex Vector Bundles

Definition 60: A **complex vector bundle** is a (real) vector bundle $E \stackrel{\pi}{\to} M$ together with a smooth family $\{J_p\}_{p \in M}$ of complex structures $J_p \in End_{\mathbb{R}}(E_p)$, $J_p^2 = -\mathrm{id}_{E_p}$, i.e. each fibre E_p is a complex vector space.

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There exist local trivializations $\Phi : \pi^{-1}(U) \to U \times \mathbb{C}^k$ such that

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Vice versa: A family of such transition functions satisfying the cocycle condition defines a complex vector bundle up to isomorphism.

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Lemma 61: Let ∇ be a connection on the complex vector bundle $E \xrightarrow{\pi} M$. Then the following conditions are equivalent:

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Such a connection is called a **complex connection**.

Complex Vector Bundles

Proof of Lemma 61:

Complex Vector Bundles

Hermitian Vector Bundles

Definition 62: (i) Let $E \xrightarrow{\pi} M$ be a complex vector bundle. A Hermitian structure on E is a smooth family $\{h_p\}_{p\in M}$ of Hermitian products on E_p, i.e. $\mathbb{R}$-bilinear forms which are $\mathbb{C}$-linear in the first and $\mathbb{C}$-antilinear in the second component, satisfy $h_p(w,v) = \overline{h_p(v,w)}$ for $v, w \in E_p$ and $h_p(v,v) > 0$ if $v \neq 0$. In particular, the real part $g = \mathrm{Re}(h)$ is a euclidean structure.

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(ii) A complex connection ∇ on an Hermitian vector bundle (E,h) is called **Hermitian** if it is metric w.r.t. g, provided J is orthogonal w.r.t. g.

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(ii) A complex connection ∇ on an Hermitian vector bundle (E, h) is called **Hermitian** if it is metric w.r.t. g, provided J is orthogonal w.r.t. g.

Remark: h is determined by g and, obviously, vice versa. We have

$$h(.,) := g(.,.) + ig(., J.)$$

(Exercise)

Hermitian Vector Bundles

Lemma 63: (i) Let (E, h) be a Hermitian vector bundle over a manifold M. Then the local trivializations $\Phi : \pi^{-1}(U) \to U \times \mathbb{C}^k$ can be chosen to be Hermitian isomorphisms.

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(iii) W.r.t. a trivialization described in (i) the connection 1–form and the curvature satisfy

$$A_k^\ell = -\overline{A_\ell^k} \quad \text{and } F_k^\ell = -\overline{F_\ell^k}.$$

Proof: Exercise

Almost Complex Structures

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Remark: (a) As seen above, (g, J) determine via $h := g + i\omega$ a Hermitian structure on TM.
(b) ω is non-degenerate: at any $p \in M$: the linear map $X \in T_pM \mapsto \omega(X, .) \in T^*M$ is an isomorphism.
Exxamples: (1) $M = \mathbb{C}^n$. Then $T_p\mathbb{C}^n \cong \mathbb{C}^n$ and for $X \in T_p\mathbb{C}^n$

$$J_p(X) := \mathrm{i}X.$$

Almost Complex Structures

Examples: (2) Let (Σ, g) be an oriented surface with a Riemannian metric g. For $X \in T_p\Sigma$ we define $J_p(X)$ by requiring, that $\{X; J_p(X)\}$ is an *oriented orthonormal basis* of $(T_p\Sigma, g_p)$. $J_p(X)$ is the counterclockwise rotated X! g defines a hermitian structure on (Σ, J).

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There exists an atlas of Σ such that for each chart (U, φ, V)

$$d\varphi \circ \mathrm{i} = J \circ d\varphi.$$

(non-trivial!). The transition maps in such atlas are holomorphic functions (exercise).

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(Σ, J) is called **Riemann surface**, J its **conformal structure**. In Algebraic Geometry, (Σ, J) is called **complex curve** if $\partial\Sigma = \emptyset$.

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