Differential Geometry II

Almost Complex Manifolds,

Klaus Mohnke

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Complex Vector Bundles

Lemma 61: Let ∇ be a connection on the complex vector bundle $E \xrightarrow{\pi} M$. Then the following conditions are equivalent:

(i) $\nabla J \equiv 0$,

(ii) For any smooth $f : M \to \mathbb{C}$ and section $\sigma : M \to E$ we have

$$\nabla(f\sigma) = df\sigma + f\nabla\sigma$$

(iii) The connection 1-form w.r.t. any complex trivialization has

the form $A \in \Omega^1(U; M(k, \mathbb{C}))$ with $M(k, \mathbb{C}) \subset M(2k; \mathbb{R})$ understood as (real) subalgebra.

Such a connection is called a **complex connection**.

Complex Vector Bundles

Proof of Lemma 61: $(i) \Rightarrow (iii)$

$\nabla J \equiv 0$. Choose cplx trivialization $\Phi : \pi^{-1}(U) \to U \times \mathbb{C}^k \simeq U \times \mathbb{R}^{2k}$, $\Phi = (\pi, \varphi)$

$$i \cdot v \mathrel{\hat=} \underbrace{\begin{pmatrix} 0 & -1 & & & & \\ 1 & 0 & & & & 0 \\ & & 0 & -1 & & \\ & & 1 & 0 & & \\ & 0 & & & \ddots & \\ & & & & & 0 & -1 \\ & & & & & 1 & 0 \end{pmatrix}}_{=: J_0} v$$

for $e \in \pi^{-1}(U)$: $\Phi(J_{\pi(e)} e) = (\pi(e), i\varphi(e))$

Let $A \in \Omega^1(U, M(2k, \mathbb{R}))$ be the corresponding connection 1 form.

mult. by i

$$\nabla J \equiv 0 \iff (d + A) \circ J_0 - J_0 (d + A) = 0$$

$$(\nabla J)\sigma = \nabla(J\sigma) - J \cdot \nabla \sigma \qquad \iff A \cdot J_0 - J_0 \cdot A = 0$$

$$\overset{\text{def. of } \mathbb{C}\text{-mult}}{\iff} A \text{ is } \mathbb{C}\text{-linear}$$

$$A((a + ib)v) = A(av + bJ(v)) = aAv + bJ_0 Av = (a + ib)Av$$

$\Rightarrow A$ is $\mathbb{C}$-linear, i.e. $A \in M(k, \mathbb{C})$

Hermitian Vector Bundles

Definition 62: (i) Let $E \xrightarrow{\pi} M$ be a complex vector bundle. A Hermitian structure on E is a smooth family $\{h_p\}_{p \in M}$ of Hermitian products on E_p, i.e. $\mathbb{R}$-bilinear forms which are $\mathbb{C}$-linear in the first and $\mathbb{C}$-antilinear in the second component, satisfy $h_p(w, v) = \overline{h_p(v, w)}$ for $v, w \in E_p$ and $h_p(v, v) > 0$ if $v \neq 0$.

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In particular, the real part, $g = \mathrm{Re}(h)$, is a euclidean structure and J is orthogonal w.r.t. g.

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(ii) A complex connection ∇ on an Hermitian vector bundle (E, h) is called **Hermitian** if it is metric w.r.t. g.

Remark: h is determined by g and, obviously, vice versa. We have

$$h(.,.) := g(.,.) + ig(., J.) \qquad (\in \mathbb{C})$$

(Exercise)

Hermitian Vector Bundles

Lemma 63: (i) Let (E, h) be a Hermitian vector bundle over a manifold M. Then the local trivializations $\Phi : \pi^{-1}(U) \to U \times \mathbb{C}^k$ can be chosen to be Hermitian isomorphisms.

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(ii) The curvature F of a Hermitian connection is skew symmetric w.r.t. h:

$$h(F\sigma, \tau) = -h(\sigma, F\tau).$$

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(ii) The curvature F of a Hermitian connection is skew symmetric w.r.t. h:

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(iii) W.r.t. a trivialization described in (i) the connection 1–form and the curvature satisfy

$$A_k^\ell = -\overline{A_\ell^k} \quad \text{and } F_k^\ell = -\overline{F_\ell^k}.$$

Proof: Exercise $\square$

Rmk: $A \in \Omega^1(\mathcal{U}, \mathfrak{u}(k))$, $F \in \Omega^2(\mathcal{U}, \mathfrak{u}(k))$

$\mathfrak{u}(\ell) = \{ B \in M(\ell; \mathbb{C}) \mid B = -\overline{B}^T \}$.

Almost Complex Structures

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Remark: (a) As seen above, (g, J) determine via $h := g - i\omega$ a Hermitian structure on TM.
(b) ω is non-degenerate: at any $p \in M$: the linear map

$$X \in T_pM \mapsto \omega(X, .) \in T^*_p M$$

is an isomorphism.
Examples: (1) $M = \mathbb{C}^n$. Then $T_p\mathbb{C}^n \cong \mathbb{C}^n$ and for $X \in T_p\mathbb{C}^n$

$$J_p(X) := iX$$

and the standard Hermitian form $\langle ., . \rangle$.

Almost Complex Structures

Examples: (2) Let (Σ, g) be an oriented surface with a Riemannian metric g. For $X \in T_p\Sigma$ we define $J_p(X)$ by requiring, that $\{X; J_p(X)\}$ is an *oriented orthonormal basis* of $(T_p\Sigma, g_p)$. $J_p(X)$ is the counterclockwise rotated X! g defines a Hermitian structure on (Σ, J).

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There exists an atlas of Σ such that for each chart (U, φ, V)

$$d\varphi \circ \mathrm{i} = J \circ d\varphi.$$

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(Σ, J) is called **Riemann surface**, J its **conformal structure**. In Algebraic Geometry, (Σ, J) is called **complex curve** if $\partial\Sigma = \emptyset$.

Kähler Manifolds

Definition 65: Let (M, J) be an almost complex manifold. The **Nijenhuis-Tensor**, N_J is the $(1, 2)$-tensor given on (local) vector fields X, Y by

$$N_J(X, Y) := [X, Y] + J[JX, Y] + J[X, JY] - [JX, JY].$$

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Proposition 66: Let (M, J, g) be a an almost Hermitian manifold, let ∇ be the Levi-Civita connection of (M, g).

∇ is a complex connection of (TM, J) if and only if $d\omega = 0$ and the Nijenhuis-tensor $N_J \equiv 0$

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Cartan's formula (see Problem Set 6) for $d\omega$ reads

$$\begin{aligned} d\omega(X, Y, Z) &= X(\omega(Y, Z)) + Y(\omega(Z, X)) + Z(\omega(X, Y)) \\ &\quad - \omega(X, [Y, Z]) - \omega(Y, [Z, X]) - \omega(Z, [X, Y]) \end{aligned}$$

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Since ∇ is torsion free we obtain

$$\begin{aligned} d\omega(X,Y,Z) &= X(\omega(Y,Z)) + Y(\omega(Z,X)) + Z(\omega(X,Y)) \\ &\quad - \omega(X, \nabla_Y Z - \nabla_Z Y) - \omega(Y, \nabla_Z X - \nabla_X Z) - \omega(Z, \nabla_X Y - \nabla_Y X) \end{aligned}$$

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Finally, last expression yields

$$(\nabla_X \omega)(Y,Z) + (\nabla_Y \omega)(Z,X) + (\nabla_Z \omega)(X,Y) = 0$$

and vanishes since $\nabla \omega \equiv 0$.

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$$\begin{aligned} &= \nabla_X Y - \nabla_Y X + J(\nabla_{JX} Y - \nabla_Y (JX)) \\ &\quad + J(\nabla_X (JY) - \nabla_{JY} X) - \nabla_{JX}(JY) + \nabla_{JY}(JX) \\ &= J((\nabla_X J)Y - (\nabla_Y J)X) - J((\nabla_{JX} J)Y - (\nabla_{JY} J)X) \\ &= 0 \end{aligned}$$

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since $\nabla J \equiv 0$ and $(\nabla J) X = \nabla (JX) - J \nabla X$.

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($\Leftarrow$) Using Koszul's formula for the Levi-Civita connection ∇

$$\begin{aligned}
&2g(\nabla_X Y, Z) \\
&\qquad = X(g(Y,Z)) + Y(g(Z,X)) - Z(g(X,Y)) \\
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one obtains a formula like

$$2g((\nabla_X J)Y, Z) = (d\omega)(X,Y,Z) \pm g(X, N_J(Y,Z)),$$

for all tangent vectors $X, Y, Z \in T_pM$ and hence

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$$\nabla J \equiv 0.$$