Differential Geometry II

Principal Fibre Bundles II

Klaus Mohnke

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Connections on Principal Fibre Bundles

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$$\Rightarrow \quad \tilde{A}_{pg} \circ d\mu_g = d\mu_g \circ \tilde{A}_p$$

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$d_e\mu(p,.) : \underline{g} \to T_p\pi^{-1}(\pi(p))$ is an isomorphism and we define

$$A_p := d_e\mu(p,.)^{-1} \circ \tilde{A}_p : T_pP \to \underline{g} = T_eG$$

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Condition (ii) translates to

$$\mathrm{Ad}_{g^{-1}} \circ A = \mu_g^* A$$

where $\alpha_g : G \to G$, $\alpha_g(h) = ghg^{-1}$ is the **conjugation**, $\mathrm{Ad}_g := d_e \alpha_g : \underline{g} \to \underline{g}$ the **adjoint representation** of G.

$\tilde{A}_p \; T_p P \longrightarrow T_p P_x \qquad x = \pi(p),\; P_x = \pi^{-1}(x)$

- linear, $\tilde{A}_p^2 = 0$ projection
- $d_p\mu_g(\mathrm{Ker}\,\tilde{A}_p) = \mathrm{Ker}\,\tilde{A}_{pg}$ (by (ii))

$\mathrm{Ker}\,\tilde{A}_p = \{v_1, \ldots, v_m\}$, $\mathrm{Im}\,\tilde{A}_p = \{w_1, \ldots, w_n\}$

bases $\quad \tilde{v}_j := d\mu_g(v_j)$, $\tilde{w}_j := d\mu_g(w_j)$

$A_{pg}\tilde{v}_j = 0$ by (ii) $\qquad \tilde{A}_p(w_j) = w_j \quad$ <u>$\tilde{A}_p$ proj</u>

- $d\mu_g(\underbrace{\mathrm{Im}\,\tilde{A}_p}_{T_p P_x}) = \mathrm{Im}\,\tilde{A}_{pg} = T_{pg} P_x$

$\Rightarrow \tilde{A}_{pg}(\tilde{w}_j) = \tilde{w}_j \quad \tilde{A}_{pg}$ proj

$\Rightarrow A_{pg} \cdot d\mu_g(v_i) = 0 = d\mu_g \cdot A_p(v_i)$

Covariant Exterior Derivative

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For a connection A we define the associated **covariant exterior derivative** $D_A : \Omega^k(M; \underline{\mathbf{g}}) \to \Omega^{k+1}(M; \underline{\mathbf{g}})$ by

$$D_A \omega := d\omega + [A, \omega]$$

Exercise: Show that $D_A \omega \in \Omega^{k+1}(M; \underline{\mathbf{g}})$.

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Definition 71: Let $A \in \mathcal{C}(P)$ be a connection. The **curvature of** A is the 2-form $F \in \Omega^2(P, \underline{g})$ given by

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Lemma 72: (i) F vanishes on tangent vectors tangent to the fibre, i.e.

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(ii) 2nd Bianchi identity: $D_A F_A = 0$.

(iii) Let $X, Y \in T_x M$ and X^h, Y^h two horizontal vector fields on P in a neighbourhood of $p \in \pi^{-1}(x)$ with $d_p\pi(X^h) = X$ and $d_p\pi(Y^h) = Y$. Then

$$F_A\widetilde{(X, Y)}_p = [X^h, Y^h]_p - [X, Y]^h_p$$

Relation to Vector Bundles

Let G be a Lie group, $P \xrightarrow{\pi} M$ be a principal G–bundle and A be a connection on P. Let $\rho : G \to Aut(V)$ be a finite-dimensional representation of G on a $\mathbb{K}$-vector space V ($\mathbb{K} = \mathbb{R}, \mathbb{C}$).

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Then A induces a connection $\nabla = \nabla^{A,\rho}$ on the associated bundle $E = P \times_\rho V$ as follows: A smooth section $\sigma : U \to E$ is uniquely determined by $\tilde{\sigma} : \pi^{-1}(U) \to V$ such that $\tilde{\sigma}(pg) = \rho(g^{-1})\tilde{\sigma}(p)$.

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$$D_A\tilde{\sigma} := d\tilde{\sigma} + \rho_*(A)(\tilde{\sigma}) \in \Omega^1(P; V)$$

vanishes on vertical tangent vectors and descends to $\nabla\sigma \in \Omega^1(M; E)$.

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The curvature of ∇ is given by

$$F^\nabla = \rho_*(F_A).$$

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$$T^h_{[p,v]} E = d_{p,v} \pi (T^h_p P)$$

for the connection on P and its induced connection on E.

The Quaternionic Hopf Bundle