

Differential Geometry II

Characteristic Classes

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The Quaternionic Hopf Bundle

Let

$$S^7 := \{A \in M(2; \mathbb{C}) \mid \text{Trace}(\bar{A}^T A) = \overset{1}{\cancel{2}}\} \subset M(2; \mathbb{C}) \cong \mathbb{C}^4.$$

$$A = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \quad \text{Trace}(\bar{A}^T A) = |a_1|^2 + |a_2|^2 + |a_3|^2 + |a_4|^2$$

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The Lie group $SU(2) \stackrel{\cong S^3}{:=} \{g \in M(2; \mathbb{C}) \mid \bar{g}^T g = \mathbf{E}_2, \det g = 1\}$ is acting on it (from the right) via $A \mapsto Ag$ and

$$S^7 / SU(2) \cong S^4 \quad \begin{aligned} &= \text{Trace}(\bar{g}^T \bar{A}^T Ag) = \text{Trace}(g^{-1} \bar{A}^T Ag) \\ &= \text{Trace}(\bar{A}^T A) = 1. \end{aligned}$$

are diffeomorphic.

Proof: $A = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \in S^7 \quad \exists g \in SU(2) : a_1 \in \mathbb{R}, a_2 = 0.$

wlog. $(a_1, a_2) \neq (0, 0)$ let $(a, b) := \frac{1}{\sqrt{|a_1|^2 + |a_2|^2}} (a_1, a_2) \quad |a|^2 + |b|^2 = 1.$

$$g := \begin{pmatrix} \bar{a} & -b \\ b & a \end{pmatrix} \quad \bar{g}^T g = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \begin{pmatrix} \bar{a} & -b \\ b & a \end{pmatrix} = \mathbf{E}_2$$

$$Ag = \begin{pmatrix} a_1 \bar{a} + a_2 \bar{b} & 0 \\ * & * \end{pmatrix} = \begin{pmatrix} \tilde{a}_{11} & 0 \\ \tilde{a}_{21} & \tilde{a}_{22} \end{pmatrix} \quad \begin{aligned} &\tilde{a}_{11} \in \mathbb{R} \quad \tilde{a}_{11}^2 + |\tilde{a}_{21}|^2 + |\tilde{a}_{22}|^2 = 1 \\ &\subset \mathbb{R} \times \mathbb{C} \times \mathbb{C} \cong \mathbb{R}^5 \end{aligned}$$

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The quotient map $S^7 \xrightarrow{\pi} S^4$ is a principal $SU(2)$ -bundle.

(Exercise)

Remark: This would be true for any free action of a compact Lie group G on a manifold M : $M \xrightarrow{\pi} M/G$ is a principal G -bundle over the manifold M/G .
Ehresmann's Theorem

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$T_p^h S^7 := (T_p \pi^{-1}([p]))^\perp$ defines a connection of the principal $SU(2)$ -bundle. (Exercise)

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Chern Classes: Axioms

Definition 73: The (rational) Chern classes are a sequence $(c_k)_{k=0}^{\infty}$ which assign to each complex vector bundle $E \xrightarrow{\pi} M$ over a smooth manifold an element $C_k(E) \in H_{DR}^{2k}(M)$ with $c_0(E) = 1$ for all E , which satisfies the following axioms:

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- (i) For all smooth $f : M \rightarrow N$: $c_k(f^*E) = f^*c_k(E)$
- (ii) $c_k(E \oplus F) = \sum_{j=0}^k c_j(E) \wedge c_{k-j}(F)$.

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(iii) For the tautological bundle $H \xrightarrow{\pi} \mathbb{C}P^1$ we have

$$c(E) = 1 - [\omega]$$

where $\omega \in \Omega^2(\mathbb{C}P^1)$ with

$$\int_{\mathbb{C}P^1} \omega = 1.$$

(iv) For $k > rk_{\mathbb{C}}(E)$ the class vanishes: $c_k(E) = 0$.

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The Proof is omitted since it should be rather discussed in the context of a lecture on Algebraic Topology or the Atiyah-Singer index theorem.

The Chern-Weil Construction

Let $(E, h) \xrightarrow{\pi} M$ be a complex vector bundle over a smooth manifold, ∇ a complex connection. Then we define the **Chern-Weil forms** $c_k(E, \nabla)$ to be the coefficients

$$\det \left(\frac{it}{2\pi} F^\nabla + \text{id}_E \right) = 1 + c_1(E, \nabla)t^2 + c_2(E, \nabla)t^4 + \dots$$

Theorem 75: (1) The Chern forms are real closed forms $c_k(E, \nabla) \in \Omega^{2k}(M; \mathbb{R})$.

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(3) The sequence c_k are the (rational) Chern classes, i.e. it satisfies the axioms.

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Lemma 76: There is a sequence of homogeneous polynomials (p_k) of degree k with rational coefficients, such that

$$c_k(E, \nabla) = p_k(s_1, \dots, s_k)$$

where

$$s_\ell = \text{Trace} \left(\left(\frac{i}{2\pi} F^\nabla \right)^\ell \right)$$

where the ℓ -th variable has degree ℓ and $p_k(0, 0, \dots, 1) \neq 0$.

The Chern-Weil Construction

Lemma 77: We have for

$\alpha \in \Omega^k(M, \text{End}(E)), \beta \in \Omega^\ell(M; \text{End}(E))$ that

$$d(\text{Trace}(\alpha \wedge \beta)) = \text{Trace}(D\alpha \wedge \beta + (-1)^k \alpha \wedge D\beta).$$

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Applying Lemma 77 inductively we conclude

$$d(s_\ell(E, \nabla)) = \text{Trace}(DF^\nabla \wedge \dots \wedge F^\nabla + \dots F^\nabla \wedge \dots \wedge DF^\nabla) = 0.$$

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Let ∇^0, ∇^1 be two complex connections on E . Then

$$\nabla^1 = \nabla^0 + \alpha$$

for $\alpha \in \Omega^1(M; \text{End}(E))$. Denote by $\nabla^t = \nabla^0 + t\alpha$ family of connections $t \in [0, 1]$. Then $F^t = F^0 + tD^0\alpha + t^2/2[\alpha, \alpha]$.

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$$\frac{dF^t}{dt} = D^0\alpha + [t\alpha, \alpha] = D^t\alpha.$$

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$$\frac{dF^t}{dt} = D^0\alpha + [t\alpha, \alpha] = D^t\alpha.$$

Hence

$$\begin{aligned} \frac{d}{dt}s_\ell(E, \nabla^t) &= \text{Trace}(D^t\alpha \wedge F^t \dots \wedge F^t + \dots) \\ &= d(\text{Trace}(\alpha \wedge F^t \wedge \dots \wedge F^t + \dots)). \end{aligned}$$

The Chern-Weil Construction

We obtain

$$s_\ell(E, \nabla^1) - s_\ell(E, \nabla^0) = d\left(\int_0^1 \text{Trace}(\alpha \wedge F^t \wedge \dots \wedge F^t + \dots)\right)$$

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