Differential Geometry II

Characteristic Classes

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The Hopf Bundle

Recall

$$S^3 := \{A(z_1, z_2) \in \mathbb{C}^2 \mid |z_1|^2 + |z_2|^2 = 1\} \subset \mathbb{C}^2.$$

The Lie group $S^1 = U(1) = \{z \in \mathbb{C} || z| = 1\}$ is acting on it (from the right) via $z \mapsto zg$. Its quotient is diffeomorphic to

$$S^3/S^1 =: \mathbb{C}P^1 \cong S^2$$

and its quotient map $S^3 \xrightarrow{\pi} S^2$ is a principal S^1-bundle.

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Trivializations are described by

$$(z_1, z_2) \in S^3 \setminus \{(z_1, z_2 \mid z_1 = 0)\} \mapsto \left(\frac{z_2|z_1|}{z_1}, \frac{z_1}{|z_1|}\right) \in B^2(1) \setminus \{(-1, 0,)\} \times S$$

and for the second coordinate likewise.

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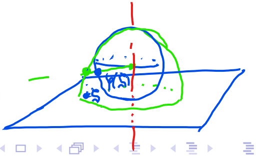
$\Phi^{-1}(\zeta, g) = (\sqrt{1-|\zeta|^2}, \zeta)\cdot g$

and for the second coordinate likewise.

B^2 is to be considered with a parametrization $\varphi : B^2 \to S^2 \setminus \{(-1, 0, 0)\}$.



The Hopf Bundle

$T^h_p S^3 := (T_p \pi^{-1}([p]))^\perp$ defines a connection A of the principal S^1–bundle. $p \in S^3$: • $d\pi_p|_{T^h_p S^3}$ is an isomorphism,

• $d_p \mu_g (T^h_p S^3) = T^h_{pg} S^3$.

The Hopf Bundle

$T_p^h S^3 := (T_p \pi^{-1}([p]))^\perp$ defines a connection A of the principal S^1–bundle.

Its curvature is described

$$\varphi^* F_A = 2 \mathrm{i} dx dy$$

Proof: $T_{(z_1,z_2)}(\pi^{-1}(\pi(z_1,z_2))) = \{ t(iz_1, iz_2) \mid t \in \mathbb{R} \}$

$\rightsquigarrow$ orthogonal projection to that

$$\tilde{A}_{(z_1,z_2)}(\underbrace{\zeta_1, \zeta_2}_{=\zeta}) = \mathrm{Re}\langle \zeta, iz \rangle iz \qquad \|iz\| = 1\ !$$

$$A_{(z_1,z_2)} = (d_e\mu((z_1,z_2), \cdot))^{-1} \tilde{A}_{(z_1,z_2)} \qquad iz = \tilde{i} \quad i \in \mathfrak{u}(1)$$

$$= \mathrm{Re}\langle \cdot, iz \rangle \cdot i$$

$x_1 + iy_1 = z_1$

$x_2 + iy_2 = z_2$

$$\Rightarrow A_{(z_1,z_2)} = i(-y_1 dx_1 + x_1 dy_1 - y_2 dx_2 + x_2 dy_2)$$

The Hopf Bundle

$$F_A = dA + \frac{1}{2}[A, A] = dA = 2i\,(dx_1 \wedge dy_1 + dx_2 \wedge dy_2)$$

$[A, A] = 0$ since S^1 is abelian

(x, y) coord. on B^2

$$(\Phi^{-1})^* F_A = (\Phi^{-1})^* (2i\, dx_2 \wedge dy_2) = 2i\, dx \wedge dy \quad \square$$

Therefore

$$\frac{i}{2\pi}\int_{S^2} F_A = \frac{i}{2\pi}\int_{B^2} \psi^* F_A = -\frac{1}{\pi}\,\mathrm{vol}(B^2) = -1.$$

$$\int_{S^2} c_1(S^3 \xrightarrow{\pi} S^2)$$

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Thus

$$c_1(S^3 \xrightarrow{\pi} S^2) = c_1(H) = \omega \in \Omega^2(S^2)$$

with

$$\int_S^2 \omega = 1.$$

The Axioms and the Chern-Weil Construction

It remains to show that the Chern-Weil forms $c_k(E, \nabla)$ defined by

$$\det(\frac{\mathrm{i}t}{2\pi} F^{\nabla} + \mathrm{id}_E) = \sum_{k=0}^{\infty} c_k(E, \nabla) t^k$$

satisfy the axioms for Chern classes.

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(i) Let $f : M \to N$ be a smooth map, $E \xrightarrow{\pi} N$ a complex vector bundle. Then the pull-back connection ∇^f is a connection on $f^*E \xrightarrow{\pi} M$ and $F^{\nabla^f} = f^* F^{\nabla}$.

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$$c_k(f^*E,\nabla^f) = f^*c_k(E,\nabla).$$

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(ii) Let $E_k \stackrel{\pi}{\to} M$, $k = 1, 2$be two complex vector bundles, ∇^k a connection on E_k. Then $\nabla := (\nabla^1, \nabla^2)^T$ defines a connection on $E_1 \oplus E_2$ and

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(iii) We computed $c_1(H) = -\omega$ above. The axiom follows with (iv)
(iv) By definition $c_\ell(E, \nabla)$ are the coefficients of the "characteristic polynomial" of the $k \times k$–matrix (in a local trivialization) $\frac{\mathrm{i}t}{2\pi} F^{\nabla}$ with entries in $\Omega^2(M, \mathbb{C})$, where $k := rk_{\mathbb{C}} E$. Hence, $c_\ell(E, \nabla) = 0$ for all $\ell > k$.

Pontrjagin Classes

Similarly, for a real vector bundle with a connection E, ∇ over a manifold M we may define $\beta_k \in \Omega^{2k}(M; \mathbb{R})$ via

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Lemma 78: (i) $[\beta_k] = 0$ for k odd.

(ii) $\left(\frac{\mathrm{i}}{2\pi}\right)^k \beta_{2k}(E,\nabla) = c_{2k}(E \otimes_{\mathbb{R}} \mathbb{C}, \nabla^{\mathbb{C}})$.

Pontrjagin Classes

Definition 79: Let $E \xrightarrow{\pi} M$ be a real vector bundle. Then

$$p_k(E) := c_{2k}(E \otimes_{\mathbb{R}} \mathbb{C}) \in H^{4k}_{DR}(M; \mathbb{R})$$

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$p_k(M)$ are another theme in the story "Curvature and Topology"!

Hirzebruch's Signature Formula

Let M be a closed oriented $4k$–manifold. Then

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is a non-degenerate symmetric bilinear form.

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In general, for a closed oriented $4k$–manifold its signature is determined by its Pontrjagin classes. The formula involves the so-called L–genus and was found by Hirzebruch.

Stiefel-Whitney Classes