Differential Geometry II

Variational Calculus

Klaus Mohnke

June 18, 2020

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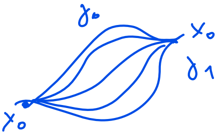
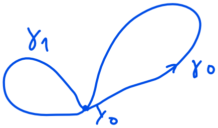
The **first Stiefel-Whitney class** is a homomorphism $w_1(E) : \pi_1(M) \to \mathbb{Z}/2\mathbb{Z}$ such that for $\gamma : [0,1] \to M$ $\gamma(0) = \gamma(1)$

$(\pi_1(M) =)\ \pi_1(M, x_0) = \{ \gamma : [0,1] \to M \mid \text{cont.}, \gamma(0) = \gamma(1) \} / \simeq$

$\gamma_0 \simeq \gamma_1 \iff \exists\, H : [0,1] \times [0,1] \to M$ cont.

$H(0, \cdot) = \gamma_0,\ H(1, \cdot) = \gamma_1,\ H(s,0) = x_0 = H(s,1)\ \forall s \in [0,1]$





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$$w_1(E)([\gamma]) = 0 \quad \Leftrightarrow \quad \exists\{\{v_j(t)\}_{j=1}^{k} \mid \text{ continuous frame of } E_{\gamma(t)}\}$$

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$\gamma^* E \to [0,1]/_{0\sim 1} \simeq S^1$

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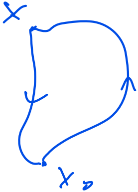

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Interpretation: Let $P \xrightarrow{\pi} M$ be the frame bundle of E. P is a principal $Gl(k;\mathbb{R})$-bundle. Let $Gl^+(k;\mathbb{R}) \subset Gl(k;\mathbb{R})$ be the subgroup of matrices with positive determinant. There exists an $Gl^+(k,\mathbb{R})$–bundle $Q \xrightarrow{\pi} M$ with a bundle map $\Phi : Q \to P$ such that $\Phi(qg) = \Phi(q)\rho(g)$ where $\rho(g) = g$ if and only if P is orientable or, equivalently, $w_1(E) = 0$.

Reductions of Principle Fibre Bundles

Q is called **reduction** of P with respect to $\rho\colon Gl^+(k;\mathbb{R}) \to Gl(k;\mathbb{R})$.

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(ii) For a complex vector bundle (E,J) the frame bundle of complex frames is the reduction of the of P w.r.t. $\rho : Gl(k;\mathbb{C}) \hookrightarrow Gl(2k;\mathbb{R})$.

(iii) For a unitary vector bundle (E,h) the frame bundle of unitary frames is the reduction of P w.r.t. $\rho : U(k) \hookrightarrow Gl(k;\mathbb{C})$.

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(iv) What additional geometric structure belongs to a reduction of the unitary frame bundle w.r.t. $\rho: SU(k)\hookrightarrow SU(k)$?

In general, ρ does not have to be injective nor injective, non-trivial $\rho: G\to G$ is possible.

The Spin groups

$\{A \in M(k,\mathbb{R}) \mid A^T A = E_k,\ \det A = 1\}$

Notice that

$$\pi_1(SO(k)) \cong \mathbb{Z}/2\mathbb{Z}.$$

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$$\widetilde{SO(k)} = \{[\gamma] | \gamma : [0,1] \to SO(k) \text{ continuous }, \gamma(0) = \mathbf{E_k}\}$$

where [.] denotes equivalence class w.r.t. homotopies fixing the end points.

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To include $k = 2$, define $Spin(2) := SO(2) = S^1$ and $\rho(g) = g^2$.

Spin Structures of Vector Bundles

Definition 80: (i) Let $(E, g) \stackrel{\pi}{\to} M$ a (real) oriented euclidean vector bundle of rank k. A **spin structure** of E is a reduction $\Phi : Q \to P$ of the corresponding principal $SO(k)$ bundle P of oriented orthogonal frames to a principal $Spin(k)$ bundle w.r.t. $\rho : Spin(k) \to SO(k)$. $\Phi(qg) = \Phi(q)\rho(g)$

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(3) The corresponding geometric structure for such a reduction is very subtle, depends on parities of the rank. It has to to with representations of the Clifford algebra of (E, g).
(4) $H^1(M; \mathbb{Z}/2\mathbb{Z}) = Hom(\pi_1(M); \mathbb{Z}/2\mathbb{Z})$ acts transitively and effectively on the set of spin structures in that is non-empty.

The 2nd Stiefel Whitney Class

Let $E \xrightarrow{\pi} M$ be a (real) oriented euclidean vector bundle of $rk(E) = k$.

$$w_2(E) \in H^2(M; \mathbb{Z}/2\mathbb{Z}).$$

That means: $\omega_2(E)$ assigns to each immersion $\varphi : \Sigma \to M$ of a closed oriented surface an element in $\mathbb{Z}/2\mathbb{Z}$ so that it is additive under disjoint unions, invariant under homotopy, zero for the constant map and if $\Phi : V \to M$ is an immersion of a compact manifold with boundary, then $w_2(\Phi|_{\partial V}) = 0$.

The 2nd Stiefel Whitney Class

% $SO(3) \simeq \mathbb{R}P^3$ ↑ S^3

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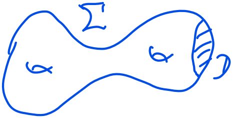
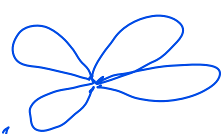
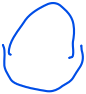
We define $w_2(E)(\varphi) = 0$ if and only if the pull-back $\varphi^* E \xrightarrow{\pi} \Sigma$ admits a spin structure.



$\Sigma \setminus D \simeq$



$D \cap \overline{\Sigma \setminus D} = S^1$



Fundamental Lemma of Calculus of Variations

$u \in L^1_{loc}$: $\forall K \subset U$ compact $u|_K \in L^1(K)$

Lemma 81: Let $u : U \subset \mathbb{R}^n \to \mathbb{R}$ be an locally integrable function on an open subset. Assume that for all smooth functions $\varphi : U \to \mathbb{R}$ with compact support in U

$$\int_U u\varphi dx = 0.$$

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Remark: We have used that in DiffGeo I for determining the critical points in the space of paths of the energy functional.

Fundamental Lemma of Calculus of Variations

Proof: Recall the cut-off function $\varphi_\epsilon : \mathbb{R}^n \to [0, 1]$ with $[0, \infty)$

$$\int_{\mathbb{R}^n} \varphi_\epsilon dx = 1 \qquad \varphi_\varepsilon \in C^\infty(\mathbb{R}^n)$$

and supp$\varphi_\epsilon \subset B_\epsilon$.

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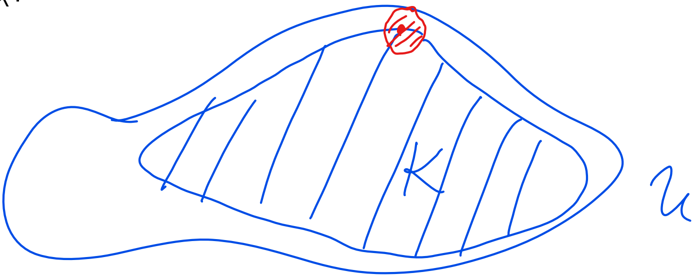

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For $\epsilon < \epsilon_K$ define the smooth function $u_\epsilon : K \to \mathbb{R}$

$$u_\epsilon(x) := \int_{\mathbb{R}^n} u(y) \underbrace{\varphi_\epsilon(y-x)}_{=0 \text{ if } y \notin B_\varepsilon(x)} dy.$$

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By assumption $u_\epsilon \equiv 0$.
On the other hand

$$\lim_{\epsilon \to 0} u_\epsilon = u|_K$$

in $L^1(K)$.

Fundamental Lemma of Calculus of Variations

Corollary: Let $\sigma : M \to E$ be a section of a vector bundle over a manifold M. Assume σ is locally integrable (e.g. σ is continuous) and that for any smooth $\varphi : M \to E^*$ with compact support

$$\int_M \langle \varphi, \sigma \rangle dM = 0.$$

M oriented & Riemannian

Then $\sigma \equiv 0$ outside a zero set.

! statement is independent of Riemannian structure & can be formulated for non-orientable M

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This is often used in the following way: Assume E is equipped with a euclidean structure and fro any smooth $\tau : M \to E$ with compact support

$$\int_M g(\varphi, \sigma) dM = 0.$$

Then $\sigma \equiv 0$ outside a zero set.

The Yang-Mills Functional

Let G be a compact Lie group, $\langle .,. \rangle$ its positive definite killing form, i.e. a scalar product on its Lie algebra $\underline{g}$ which is invariant under conjugation with an element of G.

e.g. on $\underline{o}(k) \ni X, Y$
$\langle X, Y \rangle = -\mathrm{Trace}(X \cdot Y)$

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Let $P \stackrel{\pi}{\to} M$ be a principal G–bundle over the oriented Riemannian manifold (M, g). The **Yang-Mills functional** assigns to each connection the energy of its curvature (the **field**):

$$A \in \mathcal{C}(P) \mapsto \mathcal{YM}(A) := \frac{1}{2} \int_M \|F_A\|^2 dM. \quad \in \mathbb{R}$$

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Recall that $\mathcal{C}(P)$ is an affine space over $\Omega^1(M; \underline{\mathbf{g}})$ and that

$$F_{A+\alpha} = F_A + D_A\alpha + \frac{1}{2}[\alpha, \alpha].$$

Yang-Mills Connections

We are intererested in extremal points. Necessary condition: For all $\alpha \in \Omega^1(M; \underline{\mathbf{g}})$

$$\text{"} d_A YM(\alpha) = \text{"} \quad \frac{d}{dt}\Big|_{t=0} \mathcal{YM}(A + t\alpha) = 0.$$

We have

$$\frac{d}{2dt}\Big|_{t=0} \int_M \|F_A + tD_A\alpha + \frac{t^2}{2}[\alpha, \alpha]\|^2$$

$$= \int_M \langle F_A, D_A\alpha\rangle + t\|D_A\alpha\|^2 + \frac{3t^2}{2}\langle D_A\alpha, [\alpha, \alpha]\rangle + t^3\|[\alpha, \alpha]\|^2 dM\Big|_{t=0}$$

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$$= \int_M \langle F_A, D_A\alpha\rangle dM.$$

Now the last expression is equal to

$$\int_M \langle D_A\alpha, F_A\rangle dM = \int_M \langle D_A\alpha \wedge *F_A\alpha\rangle dM$$
$$= \int_M d\langle \alpha \wedge *F_A\rangle + \langle \alpha \wedge D_A * F_A\rangle dM = \int_M \langle \alpha, D_A^* F_A\rangle dM$$

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Connections satisfying this PDE are called **Yang-Mills connections**.
How about Minima? Let $\dim M = 4$.

$$\begin{aligned} \|F_A \pm *F_A\|^2 dM &= \langle F_A \pm *F_A \wedge *(F_A \pm *F_A)\rangle \\ &= \langle F_A \pm *F_A \wedge \pm F_A + *F_A\rangle \\ &= \pm\langle F_A \wedge F_A\rangle \pm \langle *F_A \wedge *F_A\rangle + 2\langle F_A \wedge *F_A\rangle \\ &= \pm 2\langle F_A \wedge F_A\rangle + 2\|F_A\|^2 dM \end{aligned}$$

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(Anti)Self Dual Connections

We obtain

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Based on ADHM-construction and BPST-instantons one can explicitely construct all anti self dual connections on the quaternionic Hopf bundle over S^4 (see Freed,Uhlenbeck: Instantons and Four-Manifolds. Springer 1991).

Gauge Theory

Let $P \xrightarrow{\pi} M$ be a principal G-bundle for a Lie group G. Consider the associated group bundle $P \times_\alpha G = P \times G/\sim$ with the equivalence given by the G-action

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They act on the space of connections:

$$(A, g) \in \mathcal{C}(P) \times \mathcal{G}(P) \mapsto g^{-1}Ag + g^{-1}dg.$$

One studies the **moduli space of anti self dual connections**

$$\mathcal{M}(P) := \{A \in \mathcal{C}(P) \mid F_A = - * F_A\}/\mathcal{G}(P).$$

For generic Riemannian metric on M, $c_1(P) = 0$ (so-called $SU(2)$–bundle) the subspace of irreducible connections is a manifold of dimension

$$\dim \mathcal{M}^*(P) = 8c_2(P) - 3(1 - b_1(M) + b_+(M)).$$

Donaldson Theory

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Remark: In 1994 a new type of field equations, **Seiberg-Witten equations** would reprove these and often give much stronger results. However, Donaldson theory is still around...