Differential Geometry II

(Anti) Self Dual Connections and Minimal Surfaces

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The Yang-Mills Functional

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Recall that $\mathcal{C}(P)$ is an affine space over $\Omega^1(M;\underline{\mathbf{g}})$ and that

$$F_{A+\alpha} = F_A + D_A\alpha + \frac{1}{2}[\alpha,\alpha].$$

Yang-Mills Connections

We are intererested in extremal points. Necessary condition: For all $\alpha \in \Omega^1(M; \underline{\mathbf{g}})$

$$\frac{d}{dt}\Big|_{t=0} \mathcal{YM}(A + t\alpha) = 0.$$

We have

$$\begin{aligned}
&\frac{d}{2dt}\Big|_{t=0} \int_M \|F_A + tD_A\alpha + \frac{t^2}{2}[\alpha,\alpha]\|_h^2 \\
&\quad = \int_M (\langle F_A, D_A\alpha\rangle_h + t\|D_A\alpha\|_h^2 + \frac{3t^2}{2}\langle D_A\alpha, [\alpha,\alpha]\rangle_h + t^3\|[\alpha,\alpha]\|_h^2) dM\Big|_{t=0} \\
&\quad = \int_M \langle F_A, D_A\alpha\rangle_h dM.
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Variation of YM at A

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$$= \int_M \langle F_A, D_A\alpha\rangle_h dM.$$

Now the last expression is equal to

$$\int_M \langle D_A\alpha, F_A\rangle_h dM = \int_M \langle D_A\alpha \wedge *_h F_A\alpha\rangle$$

$$= \int_M d\langle \alpha \wedge *_h F_A\rangle + \langle \alpha \wedge D_A *_h F_A\rangle dM = \int_M \langle \alpha, D_A^* F_A\rangle_h dM$$

Stokes

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Connections satisfying this PDE are called **Yang-Mills connections**.

How about Minima? Let $\dim M = 4$.

$$**|_{\Omega^k} = (-1)^{k(4-k)} \cdot id$$

$$\langle *\alpha, *\beta \rangle = \langle \alpha, \beta \rangle$$

$$\begin{aligned}
\|F_A \pm *F_A\|^2 dM &= \langle (F_A \pm *F_A) \wedge *(F_A \pm *F_A) \rangle \\
&= \langle (F_A \pm *F_A) \wedge (\pm F_A + *F_A) \rangle \\
&= \pm \langle F_A \wedge F_A \rangle \pm \langle *F_A \wedge *F_A \rangle + 2 \langle F_A \wedge *F_A \rangle \\
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(Anti)Self Dual Connections

We obtain

$$\frac{1}{2}\int \|F_A\|^2 dM \geq \pm\frac{1}{2}\int_M \langle F_A \wedge F_A \rangle \quad \left(+ \int \|F_A \pm *F_A\|^2 dM \right)$$

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Based on ADHM-construction and BPST-instantons one can explicitely construct all anti self dual connections on the quaternionic Hopf bundle over S^4 (see Freed, Uhlenbeck: Instantons and Four-Manifolds. Springer 1991).

Gauge Theory

Let $P \xrightarrow{\pi} M$ be a principal G-bundle for a Lie group G. Consider the associated group bundle $P \times_\alpha G = P \times G / \sim$ with the equivalence given by the G-action

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$F_{g^*A} = g^{-1} F_A g$

They act on the space of connections:

$$(A, g) \in \mathcal{C}(P) \times \mathcal{G}(P) \mapsto g^{-1}Ag + g^{-1}dg. = g^*A$$

One studies the **moduli space of anti self dual connections**

$$\mathcal{M}(P) := \{A \in \mathcal{C}(P) \mid F_A = -*F_A\}/\mathcal{G}(P).$$

For generic Riemannian metric on M, $c_1(P) = 0$ (so-called $SU(2)$–bundles) the subspace of irreducible connections is a manifold of dimension

$$\dim \mathcal{M}^*(P) = 8c_2(P) - 3(1 - b_1(M) + b_+(M)).$$

Donaldson Theory

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Examples: (i) If the intersection form of a simply connected closed 4-manifold (i.e. the bilinear form on $H^2_{DR}(M)$ discussed earlier) is definite it is diagonalizable over $\mathbb{Z}$.

$$E_8 = \begin{pmatrix} 2 & +1 & & & & \\ +1 & 2 & & & & \\ & & \ddots & & & \\ & & & 2 & \cdot & 1 \\ & & & \cdot & 2 & 0 \\ & & & 1 & 0 & 2 \end{pmatrix} \in M(8, \mathbb{Z})$$

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Remark: In 1994 a new type of field equations, **Seiberg-Witten equations** would reprove these and often give much stronger results. However, Donaldson theory is still around...

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$$u \in \mathcal{B} := \{u : F \to \mathbb{R}^3 | u \text{ immersion, } u|_{\partial F} : \partial F \to C \text{ diffeo } \} \\ \mapsto \text{area}(u) \in (0, \infty)$$

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Proposition 82: With the notation as above

$$\frac{d}{dt}\Big|_{t=0} \text{area}(u_t) = -2 \int_F \langle X, \mathcal{H} \rangle d(u_0(F))$$

where $d(u_0(F))$ is the area measure of $u_0(F)$ and $\mathcal{H}$ its mean curvature vector.

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$$\frac{d}{dt}\Big|_{t=0} (\text{area}(u_t) - \text{area}(\tilde{u}_t)) = 0.$$

Replace u by $u = u_0 + tX$.

Minimal Surfaces

(ii) Let X_1, X_2 be two vector fields along u_0 as above. Then by chain rule

$$\frac{d}{dt}\Big|_{t=0}\text{area}(u_0+t(X_1,X_2)) = \frac{d}{dt}\Big|_{t=0}\text{area}(u_0+tX_1)+\frac{d}{dt}\Big|_{t=0}\text{area}(u_0+tX_2).$$

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(iii) Let $X_p = X_p^T + X_p^N$ such that $X_p^T = d_p u_0(\xi)$ for $\xi \in T_p F$ and $X^N \perp d_p u_0(T_p F)$ the splitting into tangent and normal part.

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(iv) Using partition of unity we xan write $X = X_1 + X_2 + ... + X_k$ where $\text{supp}(X_j) \subset U_j$ for a coordinate neighbourhood U_j of F.

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Thus, suppose $\text{supp}(X) \subset U$, (U, φ, V) coordinate chart of F. Let N be the unit normal field and

$$X = fN$$

for $f : U \to \mathbb{R}$.

Minimal Surfaces

In coordinates $(x_1, x_2) \in V$

$$\begin{aligned}
g_{ij}(t) &:= \langle \frac{\partial u_t}{\partial x_i}, \frac{\partial u_t}{\partial x_j} \rangle \\
&= g_{ij}(0) + t\Big(\frac{\partial f}{\partial x_j} \langle \frac{\partial u_0}{\partial x_i}, N \rangle + \frac{\partial f}{\partial x_i} \langle \frac{\partial u_0}{\partial x_j}, N \rangle \Big) \\
&+ tf\Big(\langle \frac{\partial u_0}{\partial x_i}, \frac{\partial N}{\partial x_j} \rangle + \langle \frac{\partial u_0}{\partial x_j}, \frac{\partial N}{\partial x_i} \rangle \Big) + O(t^2) \\
&= g_{ij}(0) - 2tfh_{ij} + O(t^2) \\
&= \sum_{k=1}^{2} (\delta_i^k - 2tfw_i^k + O(t^2)) g_{kj}.
\end{aligned}$$

where (h_{ij}) is the second fundamental form and (w_i^k) is the Weingarten map of u_0.

Minimal Surfaces

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Recall $\mathrm{Trace}(W) = 2H$.

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Recall $\mathrm{Trace}(W) = 2H$. Hence

$$\begin{aligned}
\det(g_{ij}(t)) &= \det(g_{ij}(0)(1 - 2tf\,\mathrm{Trace}(w_i^k) + O(t^2))) \\
&= \det(g_{ij}(0))(1 - 4tfH)).
\end{aligned}$$

Minimal Surfaces

Thus

$$\frac{d}{dt}\Big|_{t=0}\sqrt{\det(g_{ij}(t)} = \sqrt{\det(g_{ij}(0))}(-2fH).$$

Minimal Surfaces

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Finally,

$$\begin{aligned}\frac{d}{dt}\Big|_{t=0}\text{area}(u_t) &= -2\int_F fHd(u_0(F)) \\ &- 2\int_F \langle fN, HN\rangle d(u_0(F)) \\ &= -2\int_F \langle X, \mathcal{H}\rangle d(u_0(F)) \quad \square\end{aligned}$$