Differential Geometry II

Minimal Surfaces and Lagranian Mechanics

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Minimal Surfaces

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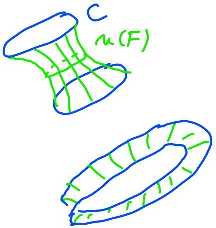
Consider the functional

$$u \in \mathcal{B} := \{u : F \to \mathbb{R}^3 | u \text{ immersion, } u|_{\partial F} : \partial F \to C \text{ diffeo } \}$$
$$\mapsto \text{area}(u) \in (0, \infty)$$



$$= \int_F d(u(F)), \quad \text{locally in coordinates}$$

$$d(u(F)) = \sqrt{\det(g_{ij}(x))}\, dx_1 dx_2$$

$$g_{ij}(x) = \left\langle \frac{\partial u}{\partial x_i}, \frac{\partial u}{\partial x_j} \right\rangle$$

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For a smooth family $\{u_t\}_{t\in(-\epsilon,\epsilon)} \subset \mathcal{B}$, $\epsilon > 0$

$$X := \frac{d}{dt}\Big|_{t=0} u_t : F \to \mathbb{R}^3$$

is a vector field along u_0 with $X_p \in T_pC$ for all $p \in \partial F$.

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Proposition 82: With the notation as above

$$\frac{d}{dt}\Big|_{t=0} \text{area}(u_t) = -2 \int_F \langle X, \mathcal{H} \rangle d(u_0(F))$$

where $d(u_0(F))$ is the area measure of $u_0(F)$ and $\mathcal{H}$ its mean curvature vector.

Define in a coord. nbhd $N = \dfrac{\frac{\partial u}{\partial x_1} \times \frac{\partial u}{\partial x_2}}{\left\| \frac{\partial u}{\partial x_1} \times \frac{\partial u}{\partial x_2} \right\|}$

$\leadsto \lambda_1, \lambda_2$ eigenvalues of Weingartenabb.

$\mathcal{H} := \left(\frac{\lambda_1 + \lambda_2}{2}\right) \cdot N$ unchanged under $N \mapsto -N$

Minimal Surfaces

In particular, if u_0 is minimal, then

$$\mathcal{H} \equiv 0.$$

(fundamental Lemma of Calculus of Variations)

Remark: Immersed surfaces with $\mathcal{H} \equiv 0$ are called **minimal surfaces** - even if they have infinite area.

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Proof: (i) Notice for all p

$$u_t(p) = u_0(p) + tX_p + O(t,p)$$

where $\partial_t O(t,p)|_{t=0} = 0$.

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For $\tilde{u}_t = u_0 + tX$ for t small

$$\frac{d}{dt}\Big|_{t=0} (\text{area}(u_t) - \text{area}(\tilde{u}_t)) = 0.$$

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Replace u by $u = u_0 + tX$.

Minimal Surfaces

(ii) Let X_1, X_2 be two vector fields along u_0 as above. Then by chain rule

$$\frac{d}{dt}\Big|_{t=0} \text{area}(u_0 + t(X_1 + X_2)) = \frac{d}{dt}\Big|_{t=0} \text{area}(u_0 + tX_1) + \frac{d}{dt}\Big|_{t=0} \text{area}(u_0 + tX_2).$$

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(iii) Let $X_p = X_p^T + X_p^N$ such that $X_p^T = d_p u_0(\xi)$ for $\xi \in T_pF$ and $X^N \perp d_p u_0(T_pF)$ the splitting into tangent and normal part.

$X_p^N = 0 \quad \forall p \in \partial F$

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and

unchanged under reparametrisation Φ_t

$$\frac{d}{dt}\Big|_{t=0}\text{area}(u_0 + tX^T) = \frac{d}{dt}\Big|_{t=0}\text{area}(u_0 \circ \Phi_t) = 0.$$

Minimal Surfaces

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(iv) Using partition of unity we xan write $X = X_1 + X_2 + ... + X_k$ where $\text{supp}(X_j) \subset U_j$ for a coordinate neighbourhood U_j of F.

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Thus, suppose $\text{supp}(X) \subset U$, (U, φ, V) coordinate chart of F. Let N be the unit normal field and

$$X = fN$$

for $f : U \to \mathbb{R}$.

Minimal Surfaces

In coordinates $(x_1, x_2) \in V$

$$
\begin{aligned}
g_{ij}(t) &:= \langle \frac{\partial u_t}{\partial x_i}, \frac{\partial u_t}{\partial x_j} \rangle \\
&= g_{ij}(0) + t\Big(\frac{\partial f}{\partial x_j} \langle \frac{\partial u_0}{\partial x_i}, N \rangle + \frac{\partial f}{\partial x_i} \langle \frac{\partial u_0}{\partial x_j}, N \rangle \Big) \\
&+ tf\Big(\langle \frac{\partial u_0}{\partial x_i}, \frac{\partial N}{\partial x_j} \rangle + \langle \frac{\partial u_0}{\partial x_j}, \frac{\partial N}{\partial x_i} \rangle \Big) + O(t^2) \\
&= g_{ij}(0) - 2tfh_{ij} + O(t^2) \\
&= \sum_{k=1}^{2} (\delta_i^k - 2tfw_i^k + O(t^2)) g_{kj}.
\end{aligned}
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where (h_{ij}) is the second fundamental form and (w_i^k) is the Weingarten map of u_0.

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Recall $\text{Trace}(W) = 2H$.

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where (h_{ij}) is the second fundamental form and (w_i^k) is the Weingarten map of u_0.

Recall $\mathrm{Trace}(W) = 2H$. Hence

$$\begin{aligned} \det(g_{ij}(t)) &= \det(g_{ij}(0)(1 - 2tf\,\mathrm{Trace}(w_i^k) + O(t^2))) \\ &= \det(g_{ij}(0))(1 - 4tfH)). + O(t^2) \end{aligned}$$

Minimal Surfaces

$\sqrt{1+t} = 1 + \frac{t}{2} + O(t^2)$

Thus

$$\frac{d}{dt}\Big|_{t=0}\sqrt{\det(g_{ij}(t))} = \sqrt{\det(g_{ij}(0))}(-2fH).$$

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$$\frac{d}{dt}\bigg|_{t=0} \sqrt{\det(g_{ij}(t)} = \sqrt{\det(g_{ij}(0))}(-2fH).$$

$$= \frac{d}{dt}\bigg|_{t=0} \int_U \sqrt{\det(g_{ij}(t))}\, dx_1 dx_2 = \int_U -2fH \underbrace{\sqrt{\det(g_{ij}(0))}\, dx_1 dx_2}_{d(u_0(F))}$$

Finally,

$$\begin{aligned} \frac{d}{dt}\bigg|_{t=0} \text{area}(u_t) &= -2\int_F fH d(u_0(F)) \\ &- 2\int_F \langle fN, HN\rangle d(u_0(F)) \\ &= -2\int_F \langle X, \mathcal{H}\rangle d(u_0(F)) \quad \square \end{aligned}$$

Lagrangian Mechanics

Let M be a smooth manifold (the **configuration space**). Let

$$L : TM \times \mathbb{R} \to \mathbb{R}$$

be a smooth function. If L is constant on $\mathbb{R}$ it is called **autonomous**.

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Examples: (i) $M = \mathbb{R}^3$, $L(x, v) = \frac{m}{2}\|v\|^2 - V(x)$. V is the **potential energy** $\frac{m}{\|v\|^2}$ is the **kinetic energy** of the system. The equation of motion is Newton's equation.

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(ii) Let (M, g) be a Riemannian manifold, $L(x, \dot{x}) := \frac{1}{2}\|\dot{x}\|^2_{g(x)}$- is the system of a free mass point: no forces are acting on it. The equation of motion is the geodesic equation.

Lagrangian Mechanics

The state of a point mass at time t_0 is dexribed by location and velocity: (x_0, v_0). Its dynamics is described as the Extrema of the Lagrange functional: $\gamma : [a, b] \to M$

$$\mathcal{L}(\gamma) := \int_a^b L(\gamma(t), \dot{\gamma}(t), t)dt.$$

among all differentiable curves $\gamma : [a, b] \to M$ with fixed endpoints $\gamma(a) = x_0$ and $\gamma(b) = x_1$.

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among all differentiable curves $\gamma : [a, b] \to M$ with fixed endpoints $\gamma(a) = x_0$ and $\gamma(b) = x_1$.

To describe its extremal points let ξ be a smooth vector field along γ which vanishes at $t = a, b$ and $\{\gamma_\tau : [a, b] \to M\}_{\tau \in (-\epsilon, \epsilon)}$ smooth family, $\gamma_\tau(a) = x_0$ and $\gamma_\tau(b) = x_1$ for all τ with

$$\frac{d}{d\tau}\Big|_{\tau=0} \gamma_\tau = \xi.$$

Lagrangian Mechanics

We need to compute the first variation.

Lemma 83: There is an smooth section $X_{L,\gamma} \in \Gamma(\gamma^* T^* M)$ such that

$$\frac{d}{d\tau}\Big|_{\tau=0}\mathcal{L}(\gamma_\tau) =: d_\gamma \mathcal{L}(\xi) = \int_a^b X_{L,\gamma}(\xi)(t)dt.$$

for all smooth vectorfields ξ along γ.

The proof usually starts with the remark that it suffices to consider ξ with support in a coordinate neighbourhood of a chart (U, φ, V) of M (as we have done for minimal surfaces). In such coordinates one shows that

$$X_{L,\gamma}(t) = \sum_{j=1}^{n} \Big(\frac{\partial L}{\partial x_j}(\gamma(t), \dot\gamma(t), t) - \frac{d}{dt}\big(\frac{\partial L}{\partial \dot x_j}(\gamma(t), \dot\gamma(t), t)\big)\Big) dx^j.$$

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Hence if γ is extremal it implies teh **Euler-Lagrange equations** which in local coordinates are given as

$$\frac{\partial L}{\partial x_j}(\gamma(t)\dot{\gamma}(t), t) - \frac{d}{dt}(\frac{\partial L}{\partial \dot{x}_j}(\gamma(t), \dot{\gamma}(t), t)) = 0$$

for all $j = 1, ..., n$.

Lagrangian Mechanics

Examples: (1) $M = \mathbb{R}^3$, $L(x, v) = \frac{m}{2}\|v\|^2 - V(x)$ $(v = \dot{x})$. The Euler-Lagrange equations boil down to

$$\frac{d}{dt}(m\dot{\gamma}) = -\nabla V(\gamma(t)),$$

Newton's equations of classical mechanics.

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(2) In local coordinates $L(x, \dot{x}) = \frac{1}{2}\|\dot{x}\|^2_{g(x)} = \frac{1}{2}\sum_{i,j=1}^n g_{ij}(x)\dot{x}^i\dot{x}^j$.

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$$\sum_{ij=1}^n \frac{\partial g_{ij}}{\partial x_k}(\gamma(t))\dot{\gamma}_i(t)\dot{\gamma}_j(t) - 2\frac{d}{dt}g_{ij}(\gamma(t))\dot{\gamma}_j(t) = 0$$

for all $k = 1, ..., n$, i.e. the geodesic equations!

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A global formulation is given by

$$(\nabla_{\dot{\gamma}}\dot{\gamma} =)\nabla^{\gamma}_{\frac{d}{dt}}\dot{\gamma} = 0.$$

The Euler-Lagrange Equations

We derive a global formulation of general Euler-Lagrange-equations. We have:

$$\begin{aligned} d_\gamma \mathcal{L}(\xi) &= \frac{d}{d\tau}\Big|_{\tau=0} \int_a^b L(\gamma_\tau(t), \dot{\gamma}_\tau(t), t)dt \\ &= \int_a^b \frac{d}{d\tau}\Big|_{\tau=0} L(\gamma_\tau(t), \dot{\gamma}_\tau(t), t)dt \\ &= \int_a^b d_{(\gamma(t),\dot{\gamma}(t))} L_t(\frac{d}{d\tau}\Big|_{\tau=0} \dot{\gamma}_\tau(t))dt. \end{aligned}$$

where $L_t : TM \times \mathbb{R} \to \mathbb{R}$ is $L_t(x, v) := L(x, v, t)$.

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&= \int_a^b d_{(\gamma(t),\dot{\gamma}(t))} L_t\Big(\frac{d}{d\tau}\Big|_{\tau=0} \dot{\gamma}_\tau(t)\Big)dt.
\end{aligned}
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where $L_t : TM \times \mathbb{R} \to \mathbb{R}$ is $L_t(x, v) := L(x, v, t)$. What is

$$
\frac{d}{d\tau}\Big|_{\tau=0} \dot{\gamma}_\tau(t) \in T_{(\gamma(t),\dot{\gamma}(t))} TM \quad ?
$$

The Euler-Lagrange Equations

We derive a global formulation of general Euler-Lagrange-equations. We have:

$$\begin{aligned} d_\gamma \mathcal{L}(\xi) &= \frac{d}{d\tau}\Big|_{\tau=0} \int_a^b L(\gamma_\tau(t), \dot{\gamma}_\tau(t), t)dt \\ &= \int_a^b \frac{d}{d\tau}\Big|_{\tau=0} L(\gamma_\tau(t), \dot{\gamma}_\tau(t), t)dt \\ &= \int_a^b d_{(\gamma(t),\dot{\gamma}(t))} L_t(\frac{d}{d\tau}\Big|_{\tau=0} \dot{\gamma}_\tau(t))dt. \end{aligned}$$

where $L_t : TM \times \mathbb{R} \to \mathbb{R}$ is $L_t(x, v) := L(x, v, t)$. What is

$$\frac{d}{d\tau}\Big|_{\tau=0} \dot{\gamma}_\tau(t) \in T_{(\gamma(t),\dot{\gamma}(t))} TM \quad ?$$

Fix a connection ∇ on $TM \xrightarrow{\pi} M$. Recall for the smooth map $\pi : TM \to M$

$$d_{(\gamma(t),\dot{\gamma}(t))} \pi(\frac{d}{d\tau}\Big|_{\tau=0} \dot{\gamma}_\tau(t)) = \frac{d}{d\tau}\Big|_{\tau=0} (\pi(\dot{\gamma}_\tau(t))) = \frac{d}{d\tau}\Big|_{\tau=0} \gamma_\tau(t) = \xi(t).$$

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With the isomorphism $(d_{(\gamma(t),\dot{\gamma}(t)}\pi)^{-1} : T_{\gamma(t)}M \to T^h_{(\gamma(t),\dot{\gamma}(t))}TM$

$$\frac{d}{d\tau}\Big|_{\tau=0}\dot{\gamma}_\tau(t) - (d_{(\gamma(t),\dot{\gamma}(t)}\pi)^{-1}(\xi(t)) = \nabla^{\gamma}_{\frac{d}{dt}}\xi(t).$$

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We get

$$\begin{aligned} & d_{(\gamma(t),\dot\gamma(t))}L_t(\frac{d}{d\tau}\Big|_{\tau=0}\dot\gamma_\tau(t)) \\ & = d_{(\gamma(t),\dot\gamma(t))}L_t(d_{(\gamma(t),\dot\gamma(t)}\pi)^{-1}(\xi(t))) + d_{(\gamma(t),\dot\gamma(t))}L_t(\nabla^\gamma_{\frac{d}{dt}}\xi(t)). \end{aligned}$$

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We identified $T_{(x,v)}(T_xM) \cong T_xM$.

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Partial integration yields

$$\int_a^b d_{(\gamma(t),\dot\gamma(t))}L_t(\nabla^\gamma_{\frac{d}{dt}}\xi(t))dt = -\int_a^b \nabla^\gamma_{\frac{d}{dt}}(d^v_{(\gamma(t),\dot\gamma(t))}L_t)(\xi(t))dt,$$

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where $d^v_{(\gamma(t),\dot\gamma(t))}L_t = d_{(\gamma(t),\dot\gamma(t))}L_t\Big|_{T_{(\gamma(t),\dot\gamma(t))}(T_{\gamma(t)}M)} \in T^*_{\gamma(t)}M$,

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We end up with

$$d_\gamma \mathcal{L}(\xi) = \int_a^b \Big(d_{(\gamma(t),\dot{\gamma}(t))} L_t \circ (d_{(\gamma(t),\dot{\gamma}(t)} \pi)^{-1} - \nabla^{\gamma}_{\frac{d}{dt}} (d^{v}_{(\gamma(t),\dot{\gamma}(t))} L_t) \Big)(\xi(t)) dt$$

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Proposition 84: An extremal path $\gamma : [a, b] \to M$ in the space of all such maps with the same endpoints $\gamma(a) = x_0$ and $\gamma(b) = x_1$ satisfies the Euler-Lagrange equations

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Remark: Notice: Both terms depend on the auxilary connection ∇ chosen, their difference, however, does not. (Exercise: Show this directly without referring to the fact that these equations describe the critical points of a functional which is defined without reference to ∇)