Differential Geometry II

Geodesics, Jacobi Fields and Conjugated Points

Klaus Mohnke

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Lagrangian Mechanics

$L : TM \times \mathbb{R}_{b} \to \mathbb{R}$ Lagrange function, $p, q \in M$, $\gamma : [a,b] \to M$, $\gamma(a) = p$, $\gamma(b) = q$

$\mathcal{L}(\gamma) = \int_a^b L(\gamma(t), \dot\gamma(t), t)\, dt$

We need to compute the first variation.

Lemma 83: There is an smooth section $X_{L,\gamma} \in \Gamma(\gamma^* T^* M)$ such that

$$d_\gamma \mathcal{L}(\xi) = \int_a^b \underline{X_{L,\gamma}(\xi)}(t) dt.$$

for all smooth vectorfields ξ along γ.
A global description is given by

$$X_{L,\gamma} = g(\nabla^\gamma_{\frac{d}{dt}} \dot\gamma, .) (= g(\nabla_{\dot\gamma} \dot\gamma, .)).$$

The Euler-Lagrange Equations

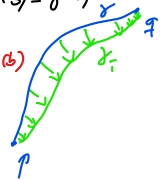
We derive a global formulation of general Euler-Lagrange-equations. We have:

$(\gamma_\tau)_{\tau \in (-\varepsilon, \varepsilon)}$ family smooth

$\gamma_\tau(a) = \gamma(a)$, $\gamma_\tau(b) = \gamma(b)$

$\frac{d}{d\tau}\big|_{\tau=0} \gamma_\tau = \xi$

$\xi(a) = 0 = \xi(b)$



$$
\begin{aligned}
d_\gamma \mathcal{L}(\xi) &= \frac{d}{d\tau}\Big|_{\tau=0} \int_a^b L(\gamma_\tau(t), \dot\gamma_\tau(t), t)\,dt \\
&= \int_a^b \frac{d}{d\tau}\Big|_{\tau=0} L(\gamma_\tau(t), \dot\gamma_\tau(t), t)\,dt \\
&= \int_a^b d_{(\gamma(t),\dot\gamma(t))} L_t\Big(\frac{d}{d\tau}\Big|_{\tau=0} \dot\gamma_\tau(t)\Big)\,dt.
\end{aligned}
$$

$\frac{d}{d\tau}\big|_{\tau=0} \dot\gamma_\tau(t) \in T_{(\gamma(t), \dot\gamma(t))}(TM)$, $d_{(\gamma(t),\dot\gamma(t))} L_t \in T^*_{(\gamma(t),\dot\gamma(t))}(TM)$

where $L_t : TM \times \mathbb{R} \to \mathbb{R}$ is $L_t(x, v) := L(x, v, t)$.

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where $L_t : TM \times \mathbb{R} \to \mathbb{R}$ is $L_t(x, v) := L(x, v, t)$. What is

$$
\frac{d}{d\tau}\Big|_{\tau=0} \dot{\gamma}_\tau(t) \in T_{(\gamma(t),\dot{\gamma}(t))} TM \quad ?
$$

$T_{(x,v)}(T_x M) = T_x M$

$\pi : (x, v) \in TM \mapsto x \in M$

$d_{(\gamma(t),\dot{\gamma}(t))}\pi\Big(\frac{d}{d\tau}\Big|_{\tau=0} \dot{\gamma}_\tau(t)\Big) = \frac{d}{d\tau}\Big|_{\tau=0} \pi(\dot{\gamma}_\tau(t)) = \frac{d}{d\tau}\Big|_{\tau=0}(\gamma_\tau(t)) = \xi(t)$

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where $L_t : TM \times \mathbb{R} \to \mathbb{R}$ is $L_t(x, v) := L(x, v, t)$. What is

$$\frac{d}{d\tau}\Big|_{\tau=0} \dot\gamma_\tau(t) \in T_{(\gamma(t),\dot\gamma(t))} TM \quad ?$$

Fix a connection ∇ on $TM \xrightarrow{\pi} M$. Recall for the smooth map $\pi : TM \to M$

$$d_{(\gamma(t),\dot\gamma(t))}\pi\Big(\frac{d}{d\tau}\Big|_{\tau=0} \dot\gamma_\tau(t)\Big) = \frac{d}{d\tau}\Big|_{\tau=0} (\pi(\dot\gamma_\tau(t))) = \frac{d}{d\tau}\Big|_{\tau=0} \gamma_\tau(t) = \xi(t).$$

The Euler-Lagrange Equations

Fix a connection on TM.
$\rightsquigarrow$ $T(TM) = T^h(TM) \oplus TM$

With the isomorphism $(d_{(\gamma(t),\dot{\gamma}(t)}\pi)^{-1} : T_{\gamma(t)}M \to T^h_{(\gamma(t),\dot{\gamma}(t))}TM$

$$\frac{d}{d\tau}\Big|_{\tau=0}\dot{\gamma}_\tau(t) - (d_{(\gamma(t),\dot{\gamma}(t)}\pi)^{-1}(\xi(t)) = \nabla^{\gamma}_{\frac{d}{dt}}\xi(t).$$

The Euler-Lagrange Equations

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We get

$$d_{(\gamma(t),\dot\gamma(t))}L_t(\frac{d}{d\tau}\Big|_{\tau=0}\dot\gamma_\tau(t))$$

$$= d_{(\gamma(t),\dot\gamma(t))}L_t(d_{(\gamma(t),\dot\gamma(t)}\pi)^{-1}(\xi(t))) + d_{(\gamma(t),\dot\gamma(t))}L_t(\nabla^\gamma_{\frac{d}{dt}}\xi(t)).$$

$\in T_{(\gamma(t),\dot\gamma(t))}(T_{\gamma(t)}M)$

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We identified $T_{(x,v)}(T_xM) \cong T_xM$.

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Partial integration yields

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where $d^v_{(\gamma(t),\dot\gamma(t))}L_t = d_{(\gamma(t),\dot\gamma(t))}L_t\Big|_{T_{(\gamma(t),\dot\gamma(t))}(T_{\gamma(t)}M)} \in T^*_{\gamma(t)}M,$

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where $d^v_{(\gamma(t),\dot\gamma(t))}L_t = d_{(\gamma(t),\dot\gamma(t))}L_t\Big|_{T_{(\gamma(t),\dot\gamma(t))}(T_{\gamma(t)}M)} \in T^*_{\gamma(t)}M$, the covariant derivative applied to it is the one induced by ∇^γ and we make use of $\xi(a) = \xi(b) = 0$.

The Euler-Lagrange Equations

We end up with

$$d_\gamma \mathcal{L}(\xi) = \int_a^b \Big(d_{(\gamma(t),\dot\gamma(t))} L_t \circ (\underbrace{d^{\nabla}_{(\gamma(t),\dot\gamma(t))}\pi)^{-1}}_{\text{depends on } \nabla} - \nabla^\gamma_{\frac{d}{dt}} (d^v_{(\gamma(t),\dot\gamma(t))} L_t) \Big)(\xi(t)) dt$$

which has to vanish for all ξ.

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Proposition 84: An extremal path $\gamma : [a, b] \to M$ in the space of all such maps with the same endpoints $\gamma(a) = x_0$ and $\gamma(b) = x_1$ satisfies the Euler-Lagrange equations

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Remark: Notice: Both terms depend on the auxilary connection ∇ chosen, their difference, however, does not. (Exercise: Show this directly without referring to the fact that these equations describe the critical points of a functional which is defined without reference to ∇)

Geodesics

- (M, g)...Riemannian manifold
- ∇...Levi-Civita connection.
- $\gamma : I \to M$...smooth curve: ∇^γ pull-back of ∇ to $\gamma^* TM$

Definition 84: γ is a **geodesic** if it satisfies

$$\nabla^\gamma \dot{\gamma} \equiv 0,$$

i.e. the velocity field is parallel along γ.

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Remark: (1) Geodesics are critical points of the Lagrangian functional on smooth paths connecting two fixed points or on loops with the Lagrange function $L : TM \to \mathbb{R}$ given by $L(x, \dot{x}) = \frac{1}{2}\|\dot{x}\|^2_{g(x)}$. The interval has to be a fixed compact interval.
(2) They are also **locally** minimizing the length of curves and the curve connecting two points of minimal length is a geodesic (and in particular smooth).

Jacobi Fields

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Need to compute the Hessian or the **second variation** of the energy functional. Let ξ, η be smooth vector fields, along γ
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$$\Gamma(\sigma, \tau, t) := \exp_{\gamma(t)}(\sigma\xi(t) + \tau\eta(t)).$$

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$\Gamma(0, 0, t) = \gamma(t), \Gamma(\sigma, \tau, a) = \gamma(a), \Gamma(\sigma, \tau, b) = \gamma(b)$

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$\Gamma(0, 0, t) = \gamma(t), \Gamma(\sigma, \tau, a) = \gamma(a), \Gamma(\sigma, \tau, b) = \gamma(b)$ and

$$\frac{\partial}{\partial\sigma}\Big|_{\sigma=0,\tau=0}\Gamma = \xi, \quad \frac{\partial}{\partial\tau}\Big|_{\sigma=0,\tau=0}\Gamma = \eta.$$

Jacobi Fields

We compute

$$
\begin{aligned}
\frac{\partial^2}{\partial\sigma\partial\tau}\mathcal{L}(\Gamma(\sigma,\tau,.)) &= \frac{\partial}{\partial\tau}\int_a^b \frac{\partial}{2\partial\sigma} g(\frac{\partial\Gamma}{\partial t},\frac{\partial\Gamma}{\partial t})dt \\
&= \frac{\partial}{\partial\tau}\int_a^b g(\nabla^{\Gamma}_{\frac{\partial}{\partial\sigma}}\dot{\Gamma},\dot{\Gamma})dt \\
&= \int_a^b \Big(g(\nabla^{\Gamma}_{\frac{\partial}{\partial\tau}}\nabla^{\Gamma}_{\frac{\partial}{\partial\sigma}}\dot{\Gamma},\dot{\Gamma}) + g(\nabla^{\Gamma}_{\frac{\partial}{\partial\sigma}}\dot{\Gamma},\nabla^{\Gamma}_{\frac{\partial}{\partial\tau}}\dot{\Gamma})\Big)dt
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We use that ∇ is torsion free and get for the first term

$$\int_a^b g(\nabla^\Gamma_{\frac{\partial}{\partial\tau}}\nabla^\Gamma_{\frac{\partial}{\partial\sigma}}\dot\Gamma,\dot\Gamma)dt = \int_a^b g(\nabla^\Gamma_{\frac{\partial}{\partial\tau}}\nabla^\Gamma_{\frac{\partial}{\partial t}}\frac{\partial\Gamma}{\partial\sigma},\dot\Gamma)dt.$$

Then with

$$R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$$

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to see that this is

$$= \int_a^b g(\nabla^\Gamma_{\frac{\partial}{\partial t}}\nabla^\Gamma_{\frac{\partial}{\partial\tau}}\frac{\partial\Gamma}{\partial\sigma},\dot\Gamma)dt + \int_a^b g(R(\frac{\partial\Gamma}{\partial\tau},\dot\Gamma)\frac{\partial\Gamma}{\partial\sigma},\dot\Gamma)dt.$$

Jacobi Fields

Now partially integrate, use that ξ, η vanish at the end points and $\nabla^\gamma \dot{\gamma} = 0$ te see that the first term vanishes. for $\sigma = \tau = 0$

$$\nabla^{\Gamma}_{\frac{\partial}{\partial t}} \dot{\Gamma} \Big|_{\sigma=\tau=0} = \nabla^{\gamma}_{\frac{d}{dt}} \dot{\gamma} = 0$$

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Using torsion-freeness again and take $\sigma = \tau = 0$ we end up with

$$\frac{\partial^2}{\partial\sigma\partial\tau}\mathcal{L}(\Gamma(\sigma,\tau,.))\Big|_{\sigma=\tau=0} = \int_a^b \Big(g(-R(\eta,\dot\gamma)\dot\gamma,\xi) + g(\nabla^\Gamma_{\frac{\partial}{\partial t}}\xi, \nabla^\Gamma_{\frac{\partial}{\partial t}}\eta)\Big) dt$$

$$= -\int_a^b g(\nabla^\Gamma_{\frac{\partial}{\partial t}}\nabla^\Gamma_{\frac{\partial}{\partial t}}\eta + R(\eta,\dot\gamma)\dot\gamma, \xi) dt.$$

Comment $g\big(R(\xi,\dot\gamma)\dot\gamma, \xi\big)(t) = K_g(\dot\xi(t), \dot\gamma(t))$

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Definition 85: Let $\gamma : I \to M$ be a geodesic. A vector field ~~ξ~~ η along γ is called **Jacobi field** if it satisfies

$$\nabla^\Gamma_{\frac{\partial}{\partial t}}\nabla^\Gamma_{\frac{\partial}{\partial t}}\eta + R(\eta,\dot{\gamma})\dot{\gamma} = 0.$$

Recall $\exp_p : U \subset T_pM \to M$, $\exp_p(X) = \gamma_X(1)$ where $\gamma_X : [0,1] \to M$ is the unique geodesic with $\gamma_X(0) = p$ and $\dot{\gamma}_X(0) = X$.

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If (M,g) is a complete metric space, then $U = T_pM$.

Conjugated Points

Proposition 86: (i) Let $\gamma : I \to M$ be a geodesic and ξ a Jacobi field along γ.
Then

$$\xi(t) = \xi_0(t) + (a + bt)\dot{\gamma}(t)$$

for a Jacobi field ξ_0 along γ with $g(\xi_0(t), \dot{\gamma}(t)) \equiv 0$.
(ii) Let $\exp_p : U \subset T_pM \to M$ be the exponential map at $p \in M$, U open starshaped Its differential

$$d_X \exp_p : T_X(T_pM) = T_pM \to T_{\exp_p(X)}M$$

can be described as follows. For $Y \in T_pM$ consider the Jacobi field η along the geodesic $\gamma_X : [0,1] \to M$, with $\gamma_X(0) = p, \dot{\gamma}(0) = X$ with initial conditions

$$\eta(0) = 0, \quad \nabla_{\dot{\gamma}}\eta(0) = Y.$$

Then

$$d_X \exp_p(Y) = \eta(1).$$

gp. do $\exp_p(X) = \gamma_X(1)$

Conjugated Points

Definition 87: $X \in T_pM$ (or $q = \exp_p(X) \in \operatorname{im}\exp_p$) is calles **conjugated to** p if $d_X \exp_p$ is not injective.