Differential Geometry II

Curvature and Global Properties of Riemannian Manifolds

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July 7, 2020

Jacobi Fields

Definition 85: Let $\gamma : I \to M$ be a geodesic. A vector field ξ along γ is called **Jacobi field** if it satisfies

$$\nabla^{\Gamma}_{\frac{\partial}{\partial t}} \nabla^{\Gamma}_{\frac{\partial}{\partial t}} \eta + R(\eta, \dot{\gamma})\dot{\gamma} = 0.$$

We had

Proposition 86: (i) Let $\gamma : I \to M$ be a geodesic and ξ a Jacobi field along γ.

Then

$$\xi(t) = \xi_0(t) + (a + bt)\dot{\gamma}(t)$$

for a Jacobi field ξ_0 along γ with $g(\xi_0(t), \dot{\gamma}(t)) \equiv 0$.

(ii) Let $\exp_p : U \subset T_pM \to M$ be the exponemntial map at $p \in M$, U open starshaped Its differential

$$d_X \exp_p : T_X(T_pM) = T_pM \to T_{\exp_p(X)}M$$

can be described as follows. For $Y \in T_p$ consider the Jacobi field η along the geodesic $\gamma_X : [0, 1] \to M$, with $\gamma_X(0) = p$ $\dot{\gamma}(0) = X$ with initial conditions

$d_X \exp_p(Y) = \eta(1)$

$$\eta(0) = 0, \quad \nabla_{\dot{\gamma}}\eta(0) = Y.$$

Conjugated Points

Definition 87: $X \in T_pM$ is called **conjugated to** p if $d_X \exp_p$ is not injective. Accordingly, a point $q = \gamma(a)$ is conjugated to a point $p = \gamma(b)$ if $q = \exp_p(X)$, $\gamma = \gamma_X$ up to translation and X is conjugated to p.

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$d_X \exp_p(\nabla_X Y) = Y(1) = 0$, $\nabla_X Y \neq 0$ since Y is non-trivial!

That means, there is a non-trivial Jacobi field Y along $\gamma_X : [0,1] \to M$ with $Y(0) = Y(1) = 0$. Hence, if q is conjugated to p along γ then p is conjugated to q along $\bar{\gamma}$ - the geodesic γ with a opposite parametrization.

In particular, if $\gamma(a)$ and $\gamma(b)$ are not conjugated along the geodesic γ than a Jacobi field along γ is uniquely determined by its values at a and b.

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Proposition 88: Let (M, g) be a Riemannian manifold with non-positive sectional curvature. Then there are no conjugate points along any geodesic.

In particular, for any $p \in M$ the differential of the exponential map $d_X \exp_p : T_pM \to T_{\exp_p}M$ is an isomorphism.

Conjugated Points

Proof: Let $\gamma : [a, b] \to M$ be a geodesic. We define the **index form** of γ on

$$\mathcal{C}(\gamma^* TM) := \{X \in C^0(\gamma^* TM) | X \text{ piecewise smooth, } X(a) = 0 = X(b)\}$$

by

$$I(X, Y) := \int_a^b (g(\nabla_{\dot\gamma} X, \nabla_{\dot\gamma} X) - g(R(X, \dot\gamma)\dot\gamma, Y))\, dt$$

Notice that by assumption

$$g(R(X, \dot\gamma)\dot\gamma, X) \leq 0$$

Hence $I(X, X) > 0$ for any X with $\nabla_{\dot\gamma} \neq 0$.
If there were $a \leq t_0 < t_1 \leq b$ and a non-trivial Jacobi field X along $\gamma|_{[t_0, t_1]}$ with $X(t_0) = X(t_1) = 0$ then we would find (continuing X by zero outside $[t_0, t_1]$)

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$$\begin{aligned} 0 < I(X, X) &= \int_{t_0}^{t_1} \left(g(\nabla_{\dot\gamma} X, \nabla_{\dot\gamma} X) + g(R(X, \dot\gamma)\dot\gamma, X) \right) dt \\ &= - \int_{t_0}^{t_1} \left(g(\nabla_{\dot\gamma}\nabla_{\dot\gamma} X + R(X, \dot\gamma)\dot\gamma, X) \right) dt = 0 \quad \square \end{aligned}$$

Hadamard Manifolds

Manifolds with non-positive sectional curvature are called **Hadamard manifolds**. Important examples are flat manifolds ($K = 0$) and hyperbolic space ($K = -1$).

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Theorem 89: Let (M, g) be a complete Riemannian manifold.
(1) Assume the sectional curvature is non-positive. Then for any point the exponential map $\exp_p : T_pM \to M$ is a covering of M. In particular, it is isomorphic to the universal covering of M and $\pi_k(M) = 0$ for any $k \geq 2$, i.e. any continuous map $u : S^k \to M$ is homotopic to a constant map. If M was simply connected, $\exp_p$ is diffeomorphism for any p.

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(2) Assume in addition that K is constant, $K \leq 0$. Then $(T_pM, \exp_p^* g)$ is isometric to the euclidean space if $K = 0$ or $(\mathbb{H}^n, \lambda^2 g_{\mathbb{H}})$ for an appropriate λ if $K < 0$. see Cheeger / Ebin

Hadamard Manifolds

Proof: (1) on T_pM define $h := \exp_p^* g$ non-degenerate since $d_X \exp_p$ is isomorphism $\forall X \in T_pM$

geodesics w.r.t. to h starting at $0 \in T_pM$ are the lines of T_pM through 0 &

for $\gamma : [0,L] \to T_pM$ $\gamma(t) = tX$: $l_h(\gamma) = l_g(\exp_p \circ \gamma)$
$= l_g(\gamma_X|[0,L]) =$
$= L \cdot \|X\|_{g_p}$

$\Rightarrow (T_pM, h)$ is complete.

(2) Notice : $\exp_p : (T_pM, h) \to (M, g)$ is local isometry : $h_X = \exp_p^* g$ by definition of h.

Exercise: If $(N,h) \xrightarrow{\Phi} (M,g)$ is a local isometry between Riemannian manifolds & if (N,h) is complete, then Φ is a covering. (See Cheeger / Ebin : Comparison Theorems...)

Hadamard Manifolds

We know: $p \in M$ & $q \in \Phi^{-1}(p)$ $\exists\, U_q \subset M$ open nbhd of p in M and $V_q \subset \mathcal{U}$ of q open nbhd. s.t. $\Phi|_{V_q}: V_q \simeq U_q$.

aim: $\exists\, W \subset M$ open nbhd. of p s.t. $\Phi^{-1}(W) = \bigsqcup_{q \in \Phi^{-1}(p)} W_q$ s.t. $\Phi|_{W_q}: W_q \to W$

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Theorem 90: Assume that (M, g) is a complete Riemannian manifold with one of the following bounds on its curvature: (i) For the sectional curvature we have

$$K \geq \frac{1}{R^2}, \text{ or}$$

(ii) the Ricci curvature satisfies

$$Ric(v, v) \geq \frac{n-1}{R^2}$$

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Then the diameter of M is bounded by

$$\text{diam}(M, g) \leq \pi R.$$

In particular, M is compact and its fundamental group is finite.

Index Lemma

We need the following Lemma on the index:

Lemma 91: Assume for a geodesic $\gamma : [a, b] \to M$ that there is no $t \in [a, b]$ such that $\gamma(t)$ is conjugated to $\gamma(a)$ along γ. Let X a piecewise smooth vector field along γ and ξ be the unique Jacobi field such that $\xi(a) = X(a) = 0$ and $\xi(b) = X(b)$. Then

$$I(\xi, \xi) \leq I(X, X)$$

and equality holds if and only if $X = \xi$.

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Let $J \neq 0$ be a Jacobi field along $\gamma|_{[0,t_0]}$ such that
$J(0) = J(t_0) = 0$ and extend it to vector field X along γ by zero.
$I(X, X) = 0$ on $[0, t]$.

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$\delta > 0$ small, so that no conjugate points on $\gamma|_{[t_0-\delta,t_0+\delta]}$. Let Z be the Jacobi field along $\gamma|_{[t_0-\delta,t_0+\delta]}$ with $Z(t_0 - \delta) = J(t_0 - \delta)$,
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$$Y(t) := \begin{cases} J(t) & \text{for } t \in [0, t_0 - \delta] \\ Z(t) & \text{for } t \in [t_0 - \delta, t_0 + \delta] \\ 0 & \text{for } t > t_0 + \delta \end{cases}$$

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Since $Y = X$ outside this interval we have $I(Y, Y) < I(X, X) = 0$ on $[0, t]$. $\square$

Proof of Bonnet's Theorem

Consider a geodesic $\gamma : [0, L] \to M$. Let X be vector field along γ, $X \perp \dot{\gamma}$ and $\nabla_{\dot{\gamma}} X = 0$. Define $Y(t) := \sin(\pi t/L)$. Then

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Assume γ has no conjugate points. Then the unique Jacobi field J with $Z(0) = Y(0) = 0$ and $Z(L) = Y(L) = 0$ (i.e. $J \equiv 0$) would satisfy

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giving a contradiction.

Hence γ has conjugate points and is not minimal. $\square$

Proof of Bonnet's Theorem

Symplectic Manifolds

Definition 90: Let M be a smooth manifold. A **symplectic structure** of **symplectic form** on M is a closed, non-degenerate 2-form $\omega \in \Omega^2(M)$, i.e. $d\omega = 0$ and for all $p \in M$

$$X \in T_pM \mapsto X \lrcorner \omega \in T_p^*M$$

is an isomorphism.

Lemma 91: (1) The non-degeneracy implies that $\dim M = 2n$ is even.

(2) It is equivalent to

$$\omega^n = \omega \wedge \ldots \wedge \omega \neq 0$$

is a volume form. In particular, M has to be oriented.

(3) If M is a closed manifold, then

$$b_2(M) := \dim H^2_{DR}(M) \geq 1.$$

Symplectic Manifolds

Examples